Carpe Diem, Carpe Worm: Context-Dependent Choice as Explained by Foraging Theory

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Abstract

Context-dependent choice phenomena, such as the decoy effect or choice overload, seem to violate the Weak Axiom of Revealed Preference. Such behavior is ecologically widespread; it occurs not just in human beings, but also in genetically distant animal species. This suggests that context-dependent choice may be a well-adapted response to a common choice problem. We create a dynamic foraging model with three key ingredients: multi-period consumption choice, uncertainty about a persistent environment, and inter-temporal substitutes. We show that “canonical” choice anomalies are features of optimal policy in such choice problems. Our model contains several standard economic problems as special cases. As a technical contribution, we modify the Koopmans intertemporal choice framework to accommodate stochastic problems.

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1. Introduction

Would you rather have $6 or a Cross pen? Would you rather have $6, the same Cross pen, or a second less attractive Cross pen? Since the second pen seems to be irrelevant, most people insist that their answers to the above two choice problems would be the same no matter which one is presented. If this is true – it does seem plausible – we can randomly assign people to these two choice problems and expect similar percentages of people in these two groups to opt for the first pen.

Simonson and Tversky (1992) ran the experiment and found the unexpected. 36% of the first group of subjects chose the first pen; 46% of the second did so (2% chose the less attractive pen). It appears that the irrelevant choice (the second pen) attracts a significant number of subjects to the choice that dominates it (the first pen). This effect is well-replicated and empirically robust. (Huber and Puto, 1983; Ariely and Wallsten, 1995; Pettibone and Wedell, 2000; Dhar and Simonson, 2003)

Context-dependent choice refers to cases when the presence of unchosen alternatives changes the relative popularity of other options. Taken naively, these phenomena falsify classical choice axioms, such as the Weak Axiom of Revealed Preference (Mas-Colell, Whinston, and Green, 1995), or the Independence of Irrelevant Alternatives (Luce, 1959).

One such phenomenon is the decoy effect, described above. The existence of a dominated alternative in the choice set increases the choice rate for the dominating alternative. This is also called the attraction effect or the asymmetric dominance effect.

Another such phenomenon is choice overload: The overabundance of choices of one type leads agents to abstain from choosing any of them. Iyengar and Lepper (2000) find that when a supermarket tasting booth presented 6 distinct flavors of jam, nearly 30% of customers made purchases; when it presented 24 flavors, only 3% did.3

There are, broadly, three classes of theories that address the problem of context-dependent choice. Psychological theories explain these phenomena as resulting from stylized non-optimizing mental processes, such as attribute-value aggregation (Tversky and Simonson, 1993), decision field theory (Roe, Busemeyer, and Townsend, 2001), leaky competing accumulators (Usher and McClelland, 2004), or focus-weighted utility (Kőszegi and Szeidl, 2013).4

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3The 6 flavors in one treatment were a subset of the 24 flavors in the other treatment.
4Other contributors include Stewart, Chater, and Brown (2006) and Guo and Holyoak
Neuroscientific theories explain these phenomena with models of biological processes in the brain, such as the firing of networks of neurons. (Soltani, De Martino, and Camerer, 2012; Louie, Khaw, and Glimcher, 2013; Webb, Glimcher, and Louie, 2014)

Market-rational theories explain context-dependence by appealing to models of consumers making Bayesian inferences about their own (imperfectly known) tastes. (Wernerfelt, 1995; Prelec, Wernerfelt, and Zettelmeyer, 1997) For instance, Kamenica (2008) formalizes game-theoretic interactions between consumers and a firm deciding on its product lines, and finds that context effects are present in perfect Bayesian equilibria with the D1 criterion (Banks and Sobel, 1987; Cho and Kreps, 1987).

However, all three classes of theories are complicated by the evidence that context-dependent choice behavior is widespread in the animal kingdom. The decoy effect in particular has been documented in animal experiments involving honey bees, gray jays, starlings, hummingbirds, and even single-celled slime molds. (Shafir, 1994; Shafir, Waite, and Smith, 2002; Bateson and Kacelnik, 1997; Bateson, Healy, and Hurly, 2002, 2003; Latty and Beekman, 2011) Market-rational theories do not address these cases, since animals do not ordinarily engage in consumer markets.  

These experiments also raise questions for psychological and neuroscientific theories, since the cognitive structures of these animals differ substantially from those of human beings. Even if psychological or neurobiological processes are the proximate causes of context-dependent choice, this raises a further puzzle: If context-dependent choice is a costly and inefficient behavior, why does it occur in species that are distant on the evolutionary tree?

When phylogenetically distant species exhibit similar behavior, this suggests that the behavior is a well-adapted response to a common problem. We propose that context-dependent choice is a well-adapted response to the problem of foraging; that is, the problem of making repeated consumption choices in an uncertain environment. This problem is shared by most animal species. It also contains some standard economic decisions as special cases.

In our model, an agent (a consumer shopping for goods, an animal foraging for food) makes consumption choices over multiple periods. Its key
ingredients are that the environment is uncertain but persistent, and that goods are intertemporal substitutes. We show that the optimal solution to this problem involves context-dependent behavior, such as the decoy effect and choice overload.

The intuition, for the case of the decoy effect, is as follows: Suppose that in each period the agent chooses between goods of type $X$ and some outside option. Suppose that, when faced with a medium-quality $X$ and the outside option, the agent chooses the outside option. Suppose that the agent instead encounters a medium-quality $X$, a low-quality $X$, and the outside option. If the agent is uncertain about the quality of $X$ in this environment, then (since the environment is persistent) a low-quality $X$ is generally bad news about the quality of $X$ in future. This lowers expected $X$-consumption in future, which, by intertemporal substitution, increases the value of $X$-consumption today. Thus, the agent switches from choosing the outside option to choosing the medium-quality $X$.

With our model, we show that this intuition holds formally. If, ceteris paribus, the quality of some $X$ is lower today, it is intuitive to infer that the quality of $X$ will be lower tomorrow. It turns out that, for this natural inference to hold in general, the quality of $X$ in each period must exhibit the monotone likelihood ratio property (MLRP) (Milgrom, 1981) with respect to some persistent state variable. We make this assumption. Milgrom and Yu (2014) show that this can be interpreted as the agent regarding the environment as maximally entropic, in an information theoretic sense.

We also consider the case where the agent is uncertain, not about the quality of $X$, but about the number of $X$-encounters she faces in each period. With an equivalent MLRP assumption, this produces the choice overload phenomenon. Thus, our model predicts, as a function of the uncertainty in the environment, which kinds of context-dependence agents will exhibit.

We assume that agents' preferences are dynamically consistent. That is, if two consumption plans are identical up to time $t$, and diverge only at period $t + 1$, then the agent’s preference over those plans at $t$ is the same as his preference at $t + 1$. This is not because we believe dynamic inconsistency is an unimportant behavioral phenomenon, but because a full treatment of dynamic inconsistency would distract from our main point and complicate the statement of key theorems. Informulating these preferences, we modify the Koopmans framework for intertemporal choice (Koopmans, 1960) to encompass the case of stochastic choice problems. This is a new technical contribution.
Our assumptions on preferences are, in a sense, maximally general, subject to the constraints of being dynamically consistent and exhibiting intertemporal substitution. Appendices A and B demonstrate this. Consequently, our model contains several standard economic environments as special cases, including multi-period search by a consumer with unit demand (section 5), and cake-eating problems with CARA utility (appendix C).

Our results thus suggest that context-dependent choice is a well-adapted response to a simple but ecologically widespread problem. Moreover, context dependent choice may be welfare-enhancing for consumers. While we do not assert that honeybees and slime molds are making complex dynamic optimizations in real time, our theory complements existing theories by explaining why such behavior persists in many competitive environments.

The next section introduces the general framework, which is inspired by optimal foraging theory (Stephens, 1986) but aims at broad applicability. The attraction effect is the focus of Section 3, while choice overload is the focus of Section 4. Section 5 demonstrates how the model also sheds light on multi-period search with unit demand.

2. The model

An organism simultaneously encounters multiple prey at time 1, . . . , T. For each t, \( N_t \) is the number of prey choices, and \( A_t \equiv \{0, q_{t1}, \ldots, q_{tN_t}\} \) is the choice set, where 0 means abstaining and \( q_{tn} \in (0, Q] \) represents the quality of the \( n \)-th choice. (For the motivating experiment, 0 corresponds to the $6 option, a result of the normalization.) The organism can opt for one and only one of the \( A_t \) elements. In the foraging literature, quality usually corresponds to the (expected) net energy gain from pursuing the prey. We interpret \( q_{tn} \) broadly – it may take into account the nutritional value, the handling and digesting times, and the survival and reproductive costs.

At time \( t \), the organism assigns to consumption sequence \((q^*_t, \ldots, q^*_T)\) the decision utility

\[
U_t(q^*_t, \ldots, q^*_T) \equiv u_t(q^*_t) + \delta_t(q^*_t)u_{t+1}(q^*_t) + \ldots + (\Pi_{\tau=t}^{T-1}\delta_\tau(q^*_\tau))u_T(q^*_T),
\]

where for \( \tau = 1, \ldots, T \), \( u_\tau(q^*_\tau) \geq 0 \) is (weakly) increasing in \( q^*_\tau \). We assume the organism to be an expected utility maximizer. For a result in appendix B, we need \( U_T(q^*_T) \neq U_T(q_T) \) for some \((q^*_T, q_T)\) pair.

This is a Koopmans utility setup (Koopmans, 1960; Koopmans, Diamond, and Williamson, 1964), which, in the finite horizon case, requires that for each
The decision maker maximizes $U_t(q_t^*, \ldots, q_T^*) \equiv W_t(q_t^*, U_{t+1}(q_{t+1}^*, \ldots, q_T^*))$ for some $W_t$. Specifically, we have

$$U_t(q_t^*, \ldots, q_T^*) = u_t(q_t^*) + \delta_t(q_t^*) U_{t+1}(q_{t+1}^*, \ldots, q_T^*)$$

for $t = 1, \ldots, T - 1$. By making $U_{t+1}$ enter $U_t$ linearly, we ensure that the organism has no incentive to deviate from an optimal policy that maximizes the expectation of $U_t(q_t^*, \ldots, q_T^*)$. We demonstrate that the converse is also true in appendix A, that is, the linear Koopmans setup is necessary to avoid the complications caused by time inconsistent preferences and the ensuing predicament of not being able to apply methods of dynamic programming and optimal control (Kydland and Prescott, 1977).

The setup leaves open the possibility that the organism optimally opts for a lower quality choice $q_t'$ in spite of the presence of a higher quality one $q_t > q_t'$ when $u_t(q_t) - u_t(q_t')$ is small relative to $\delta_t(q_t') - \delta_t(q_t)$ and future utility. This is rarely observed in nature. Two factors can explain this rarity: first, prey is usually freely disposable; second, discounts may not be strongly responsive to quality differences. We therefore impose a technical assumption.

**Assumption 1.** Whenever $q > q' > 0$, $u_t(q) + \delta_t(q)V_{t+1}^* \geq u_t(q') + \delta_t(q')V_{t+1}^*$, where $V_{t+1}^*$ is the highest possible continuation value.

Consequently, if the organism opts for a nonzero choice, its favorite is the choice of highest quality. This is trivially true when $\delta_t(q)$ is constant for each $q > 0$.

Appendix B demonstrates that given the above assumptions, $U_t$ functions are supermodular or submodular (Topkis, 1998; Milgrom and Shannon, 1994) if and only if $\delta_t$ functions are increasing or decreasing. It is well-known that supermodularity and submodularity corresponds to the economic concepts of complement and substitute respectively (Bulow, Geanakoplos, and Klemperer, 1985; Milgrom, 2004). Founded on introspection and observations concerning the choice problems we are investigating, we require $\delta_t(q_t^*)$ to be decreasing in $q_t^*$ for all $t$, essentially assuming prey consumptions to be intertemporal substitutes.

In summary, there are two sources of discounting in the model: impatience (Fisher, 1930) and the intertemporal substitution effect. To illustrate, we
can make $\delta_t(q) \equiv \alpha_t - \beta_t I(q) - \gamma_t(q)$, where $\alpha_t$ and $\beta_t$ are constants, $I$ is the indicator function of whether $q$ is positive, and $\gamma_t(q)$ is an increasing function. The first term models impatience; the second the substitution effect of nonzero consumption; the third the extra substitution effects of different qualities. The tradeoff is clear: a nonzero $q^*_t$ brings about an instant gain of $u_t(q^*_t)$, but also an ex post loss of $(\delta_t(0) - \delta_t(q^*_t))U_{t+1}(q^*_{t+1}, \ldots, q^*_T)$. When the organism expects high $U_{t+1}(q^*_{t+1}, \ldots, q^*_T)$, it may consider forgoing $u_t(q^*_t)$.

Appendix C explains how this utility setup emerges from classic consumption planning problems subject to budget constraints.

As for the stochastic foraging environment, an unknown parameter $\theta \in \Theta \subset \mathbb{R}$ describes how favorable the environment is. A family of conditional generalized probability density function (gpdf) $\{f(\cdot|\theta) : \theta \in \Theta\}$ satisfies the monotone likelihood ratio property (MLRP), i.e., for all $\tilde{\theta} > \theta$, $s > s'$, $f(s|\tilde{\theta})f(s'|\theta) \geq f(s|\theta)f(s'|\tilde{\theta})$. We denote a corresponding cumulative distribution function (cdf) $F(\cdot|\theta)$. These concepts are well investigated by Milgrom (1981), which serves as our main reference for mathematical results. Milgrom and Yu (2014) lend support to the MLRP assumption by showing that the maximal entropy principle, an axiomatized method of constructing probability distributions under uncertainty, entails MLRP.

Milgrom (1981) defines signals $s$ to be more favorable than $s'$ if for every prior cdf $G$ for $\theta$, the posterior $G(\cdot|s)$ (first-order) stochastically dominates $G(\cdot|s')$, i.e., $G(\theta|s) \leq G(\theta|s')$ for all $\theta \in \Theta$. It is shown that given the MLRP assumption, $s \geq s'$ implies that $s$ is more favorable than $s'$. For expository simplicity, we assume that there is a neutral signal $s^n$ for $\theta$, meaning that for every $\theta$ and $\tilde{\theta}$, $f(s^n|\theta) = f(s^n|\tilde{\theta})$. For any prior about $\theta$, observing $s^n$ results in a posterior identical to the prior. Accordingly, $s$ is said to be good news for $\theta$ if it is more favorable than neutral news; bad news for $\theta$ if it is less favorable. In our case, any $s \geq s^n$ is good news and $s \leq s^n$ is bad news.

Our models of two context-dependent choice effects differ in what this probabilistic structure describes, but the following notational convention applies to both. Given prior $G$ about $\theta$, observing choice set $A$ results in the posterior $G_A$; observing $A^1, \ldots, A^I$ results in $G_{(A^1, \ldots, A^I)}$. Let us denote for

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8 A gpdf is a probability density function (pdf) for continuous distributions and a probability mass function (pmf) for discrete distributions (DeGroot, 2005).

9 Milgrom (1981) works with strict dominance, but the weak version better fits into our paper. Shaked and Shanthikumar (1994, 2007) are standard references for stochastic orders.
each $t = 1, \ldots, T$, 

$$V_t(G) \equiv \sup_{\pi_t} \mathbb{E}[U_t(q^*_t, \ldots, q^*_T)],$$

where $\pi_t$ represents an arbitrary policy specifying how to make choices at time $t, \ldots, T$. In words, $V_t$ is the optimal value function right before observing $A_t$.

We can define the decision value of $q_{tn}$ as $[u_t(q_{tn}) + \delta_t(q_{tn})V_{t+1}(G_{A_t})] - V_{t+1}(G_{A_t})$, where the first term is the expected utility of choosing $q_{tn}$ and then applying an optimal strategy, and the second is that of not choosing any prey and then applying an optimal strategy. The decision value of abstaining is 0. It is optimal for the organism to choose the option with the highest decision value.

On account of assumption 1 that simplifies time-$t$ decision to be between 0 and $\bar{q}_t \equiv \max A_t$, we can write the Bellman equation as

$$V_t(G) = \mathbb{E}[\max\{V_{t+1}(G_{A_t}), u_t(\bar{q}_t) + \delta_t(\bar{q}_t)V_{t+1}(G_{A_t})\}].$$

Now we make the specifications concrete.

3. Attraction effect

We assume that each $N_t$ independently follows a distribution with probability mass function (pmf) $h_t(\cdot)$, and that given $\theta \in \mathbb{R}$ and $N_t$, $q_{tn}$ independently follows gpdf $f(\cdot|\theta)$. The prior of $\theta$ is $G_0$. The higher $\theta$ is, the more likely the organism encounters high quality prey.

**Proposition 3.1.** The decision value of $q_{tn} > 0$ increases when bad news $q^b$ for $\theta$ is added to $A_t$.

**Proof.** The proof is given in appendix D.

In nature, proposition 3.1 may become the pattern that a dominated choice serves as a stimulus for being attracted to the choices that dominate it.

**Corollary 3.2** (attraction effect). Suppose choosing $\bar{q}_t > 0$ out of $A_t$ is optimal. When bad news $q^b < \bar{q}_t$ for $\theta$ is added to $A_t$, choosing $\bar{q}_t$ is still optimal.

**Proof.** When the organism faces $A_t$, $\bar{q}_t$ being chosen implies that its decision value is nonnegative. Adding $q^b$ only increases its decision value, so choosing it is still no worse than abstaining.
4. Choice overload

We assume that given $\theta$, each $N_t$ independently follows a distribution with pmf $f_t(\cdot|\theta)$, and that given $N_t$, $q_{tn}$ independently follows a distribution with gpdf $h(\cdot)$. The prior of $\theta$ is $G_0$. The higher $\theta$ is, the more likely the organism encounters a large number of choices.

**Proposition 4.1.** The decision value of $q_{tn} > 0$ decreases when some $q$ is added to $A_t$.

*Proof.* The proof is given in appendix E.

By induction, when more than one $q$ is added to $A_t$, the decision value of $q_{tn}$ can only decrease.

**Corollary 4.2** (choice overload). *Suppose choosing $\bar{q}_t > 0$ out of $A_t$ is suboptimal. When some $q < \bar{q}_t$ is added to $A_t$, choosing $\bar{q}_t$ is still suboptimal.*

*Proof.* When the organism faces $A_t$, $\bar{q}_t$ has a negative decision value given that choosing it is suboptimal. When $q$ is added, by proposition 4.1, the decision value decreases and remains negative, so abstaining is still better.

When one or more $q$ are added to $A_t$ without improving the quality of the best choice, the appeal of the best choice can only decrease. The organism may opt for the best choice, but in a parallel universe that offers a larger choice set, it may abstain.

5. Application in consumer theory: multi-period search with unit demand

Imagine a shopper with a unit demand for some good and budget $Q$. She visits an online deal website at time $t = 1, \ldots, T$, and the set of deals at time $t$ that meet her budget is $\{p_{t1}, \ldots, p_{tN_t}\}$; where each $p_{tn}$ represents the price of the $n$-th deal. We can transform her choice set to be $A_t = \{0, q_{t1}, \ldots, q_{tN_t}\}$, where 0 still means abstaining and each $q_{tn} = Q - p_{tn}$, that is, the quality of $n$-th deal is its net saving for the shopper. In this way, our foraging model can shed lights on how shoppers make these economically significant decisions.

Specifically, the unit demand assumption requires $\delta_t(q) = 0$ for each $q > 0$ and $t = 1, \ldots, T-1$, meaning that the game ends once the shopper makes the purchase. Since the assumptions in section 2 are satisfied, all the following findings remain valid.
6. Discussion

Sections 3 and 4 give opposite predictions when an inferior choice is added to the choice set. In reality, the decoy effect and choice overload can coexist, and which one dominates is an empirical question. In our view, this explains why Soltani, De Martino, and Camerer (2012) find that the attraction effect is statistically significant only when decoys are sufficiently bad, and why Scheibehenne, Greifeneder, and Todd (2010) document substantial variance in the results of choice overload experiments.

Our model makes falsifiable predictions about which effect will occur as a function of the underlying environment. It also makes falsifiable predictions about how agents’ beliefs about future choice sets will relate to their current choice behavior.\footnote{These beliefs have the usual interpretation as preferences over appropriate contingent lotteries.}

Our theory is not a substitute for psychological or neuroscientific theories of context-dependent choice. Instead, it may explain why the mechanisms adduced by such theories are not eliminated by market competition or natural selection.

Context-dependent choice behavior is observed even in single-period choice experiments, many of which we cite in our introduction. This behavior could be due to factors such as:

1. Subjects making inferences due to a mistaken belief about future choice sets in the same experiment
2. Subjects making inferences about choice sets in future experiments by the same experimenter
3. Hard-wired neurological mechanisms supported by natural selection
4. Well-functioning heuristics imported from ordinary consumer interactions.

It is an open question which, if any, of these factors explain subject behavior in single-period experiments. However, our results suggest that experimenters seeking to isolate psychological or neurological context effects should emphasize to subjects the one-shot nature of their choice, and employ protocols that rule out intertemporal inference as a confound.\footnote{This may be infeasible in animal experiments.}
References


Appendix

A. Strong time consistency of preferences and the linear Koopmans setup

The utility setup of this paper is in a sense maximally general for applying the control theory to the analysis (Kydland and Prescott, 1977).

Let us assume that the organism adopts the decision utility function $U_t(q_t, \ldots, q_T)$ at each time $t$. For this section, the consumption streams can belong to arbitrary sets.

Following Siniscalchi (2011), the preference relations represented by $U_t$ functions are said to be time consistent if for $t = 1, \ldots, T - 1$,

$$U_t(q, q_{t+1} \ldots, q_T) \geq (>) U_t(q, q_{t+1}^\prime \ldots, q_T^\prime)$$

for some $q$ implies

$$U_{t+1}(q_{t+1}, \ldots, q_T) \geq (>) U_{t+1}(q_{t+1}^\prime, \ldots, q_T^\prime).$$

For example, if the organism weakly prefers $(q, q_{t+1} \ldots, q_T)$ to $(q, q_{t+1}^\prime \ldots, q_T^\prime)$ at time $t$, it weakly prefers $(q_{t+1} \ldots, q_T)$ to $(q_{t+1}^\prime \ldots, q_T^\prime)$ at time $t + 1$. Otherwise, a naif (O’Donoghue and Rabin, 1999), who solves for an optimal plan
at time \( t \) and then re-optimize at time \( t + 1 \), can initially plan to choose \((q, q_{t+1}, \ldots, q_T)\), but switch to \((q'_{t+1}, \ldots, q'_T)\) at time \( t + 1 \). If the preference relations are time consistent, the organism never has a strict incentive to renege on earlier optimal plans in a deterministic environment.

Under the framework of Galperti and Strulovici (2014), time consistency is equivalent to the Koopmans setup, i.e., for all \( t \), \( U_t(q_t, \ldots, q_T) \equiv W_t(q_t, U_{t+1}(q_{t+1}, \ldots, q_T)) \) for some \( W_t \). The Koopmans setup, however, does not guarantee that a naif always sticks to an optimal plan in a stochastic environment. Instead, we need a stronger notion of time consistency, which underlies classical studies of Strotz (1955) and Kydland and Prescott (1977).

Specifically, the preference relations represented by \( U_t \) functions are said to be strongly time consistent\(^\text{12}\) if for \( t = 1, \ldots, T - 1 \) and any distributions \( G \) and \( H \) of \((q_{t+1}, \ldots, q_T)\),

\[
\mathbb{E}_G U_t(q, q_{t+1} \ldots, q_T) \geq (>) \mathbb{E}_H U_t(q, q_{t+1} \ldots, q_T)
\]

for some \( q \) implies

\[
\mathbb{E}_G U_{t+1}(q_{t+1}, \ldots, q_T) \geq (>) \mathbb{E}_H U_{t+1}(q_{t+1}, \ldots, q_T).
\]

If the preference relations are strongly time consistent, the organism never has a strict incentive to renege on earlier optimal plans. The following result further claims that if we demand that the organism never renege, \( U_{t+1} \) enters \( U_t \) linearly.

**Proposition A.1.** In the Koopmans setup, the preference relations represented by \( U_t \) functions are strongly time consistent if and only if for all \( t \), \( U_t(q_t, \ldots, q_T) = u_t(q_t) + \delta_t(q_t) U_{t+1}(q_{t+1}, \ldots, q_T) \) for some \( u_t \) and \( \delta_t \).

**Proof.** The “if” part follows from the linearity of expectations.

To show the converse, we suppose \( W_t(q_t, U_{t+1}(q_{t+1}, \ldots, q_T)) \) is not linear in \( U_{t+1} \), that is, there exist \( \Pi_0, \Pi_1 \), and \( \Pi_2 \) in the range of \( U_{t+1} \) and \( q \) such that \( \Pi_0 = \alpha \Pi_1 + (1 - \alpha) \Pi_2 \) for some \( 0 < \alpha < 1 \) and \( W_t(q, \Pi_0) \neq \alpha W_t(q, \Pi_1) + (1 - \alpha) W_t(q, \Pi_2) \). Without loss of generality, let \( W_t(q, \Pi_0) > \alpha W_t(q, \Pi_1) + (1 - \alpha) W_t(q, \Pi_2) \). Let distribution \( G \) place unit probability mass on a consumption stream that yields \( \Pi_0 \), and \( H \) place \( \alpha \) for \( \Pi_1 \) and \( 1 - \alpha \) for \( \Pi_2 \). We have \( \mathbb{E}_G U_t(q, q_{t+1} \ldots, q_T) > \mathbb{E}_H U_t(q, q_{t+1} \ldots, q_T) \) and

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\(^{12}\)Ilya Segal suggested “backward stationarity,” a term more in line with earlier literature.
\( \mathbb{E}_G U_{t+1}(q_{t+1}, \ldots, q_T) = \mathbb{E}_H U_{t+1}(q_{t+1}, \ldots, q_T) \), a violation of strong time consistency.

The intuition for the second part of the proof can be illustrated with \( t = 1 \) and \( T = 2 \). Supposing that \( W_1 \) is nonlinear in \( U_2 \), we can construct a stochastic environment that a naif reneges on a time-1 optimal plan. To see this, let \( U_2(q^0) = \Pi_0, U_2(q^1) = \Pi_1, U_2(q^2) = \Pi_2, \Pi_0 = \alpha \Pi_1 + (1 - \alpha) \Pi_2 \), and \( W_1(q, \Pi_0) > \alpha W_1(q, \Pi_1) + (1 - \alpha) W_1(q, \Pi_2) \). The organism has to pick \( q \) at time 1, but at time 2 has the freedom to choose between \( q^0 \) or a randomization between \( q^1 \) and \( q^2 \) with weights \( \alpha \) and \( 1 - \alpha \) respectively. The only optimal plan at time 1 is to choose \((q, q^0)\), but when time 2 comes, it is also optimal to randomize.

It is worth noting that the concept of strong time consistency entails linearity in a similar way to the workings of the independence axiom of Savage (1954) and assumptions 2 and 3 of Epstein (1983). Nonetheless, our concept has a direct interpretation in terms of the consistency of dynamic choice behaviors.

Since the focus of the paper is not dynamic consistency of the organism’s choices, according to proposition A.1, the assumption of \( U_t(q_t, \ldots, q_T) = u_t(q_t) + \delta_t(q_t)U_{t+1}(q_{t+1}, \ldots, q_T) \) is necessary for us to avoid the distraction.

### B. The super/sub-modularity of \( U_t \) functions and the monotonicity of \( \delta_t \) functions

According to the definition of Topkis (1998), \( U_t \) is supermodular if \( U_t \) has increasing differences in each \((q_i, q_j)\) pair; it is submodular if \(-U_t\) is supermodular. These concepts are closely related to those of complement and substitute in the theories of consumer and producer (Bulow et al., 1985; Milgrom and Shannon, 1994). Given the linear Koopmans setup, we can establish that they corresponds to the monotonicity of \( \delta_t \) functions.

**Proposition B.1.** \( U_t \) functions are supermodular if and only if \( \delta_t \) functions are increasing.

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13 We thank Bruno Strulovici for pointing our the connections.
14 A real-valued function \( U \) on \( \mathbb{R}^T \) has increasing differences in \((q_i, q_j)\) if whenever \( q_j > q'_j \), \( f(q_1, \ldots, q_i, \ldots, q_j, \ldots, x_T) - f(q_1, \ldots, q_i, \ldots, q'_j, \ldots, x_T) \) is increasing in \( q_i \).
Proof. For \( q_j > q_j' \), we have

\[
U_t(q_t, \ldots, q_i, \ldots, q_j, \ldots, q_T) - U_t(q_t, \ldots, q_i, \ldots, q_j', \ldots, q_T) = \\
(\prod_{r=t}^{j-1} \delta_r(q_r))\{u_j(q_j) - u_j(q_j') + \left(\delta_j(q_j) - \delta_j(q_j')\right)\sum_{r=j+1}^T u_r(q_r)\}. \tag{B.1}
\]

For the “if” part, note that the term in the curly bracket is nonnegative, because it is assumed that \( u_j(q_j) \geq u_j(q_j') \) and \( \delta_j(q_j) \geq \delta_j(q_j') \). Hence, the difference is increasing in \( q_i \), meeting the definition of supermodularity.

For the “only if” part, we only need to use the increasing differences of \( U_t \) in \((q_i, q_T)\). By setting \( j = T \), we simplify equation B.1 into

\[
U_t(q_t, \ldots, q_T) - U_t(q_t, \ldots, q_T') = (\prod_{r=t}^{T-1} \delta_r(q_r))(u_T(q_T) - u_T(q_T')).
\]

Supermodularity requires this difference to be increasing in \( q_i \). Since \( u_T(q_T) \neq u_T(q_T') \) for some \((q_T, q_T')\) pair, we cannot have \( \delta_t \) strictly decreasing for any two values.

\textbf{Proposition B.2.} \( U_t \) functions are submodular only if \( \delta_t \) functions are decreasing; the “if” claim is true given assumption 1.

\textit{Proof.} The same proof strategy works. For the “if” part, assumption 1 guarantees the nonnegativity of the term in the curly bracket.

\textbf{C. Linear Koopmans setup in classic consumption planning problems}

Let us consider a classic consumption planning problem of choosing consumption sequence \((c_1, \ldots, c_T)\) to maximize \( W(c_1, \ldots, c_T) \equiv \sum_{t=1}^T \beta^{t-1} u(c_t) \) subject to budget constraint \( \sum_{t=1}^T c_t \leq B \). We can interpret the problem as an agent deciding on how to eat a cake of size \( B \) over time.

The budget constraint renders consumptions in different periods substitutes. Indeed, we can transform the problem into one that is consistent with the utility setup of our main model when \( u \) exhibit either constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA) (Arrow, 1965; Pratt, 1964).

When \( u(c) = c^\gamma \) with \( 0 < \gamma \leq 1 \) (a CRRA case), for each \( t \), we can define \( q_t \) as the proportion of remaining budget consumed in period \( t \), that
is, \( c_1 = Bq_1, c_2 = Bq_2(1 - q_1), c_3 = Bq_3(1 - q_1)(1 - q_2), \) and so on. As a result, the objective is choosing \((q_1, \ldots, q_T)\) to maximize

\[
U(q_1, \ldots, q_T) = W(c_1, \ldots, c_T) = (Bq_1)\gamma + \beta[Bq_2(1 - q_1)]\gamma + \ldots + \beta^{T-1}[Bq_T\Pi_{i=1}^{T-1}(1 - q_i)]\gamma.
\]

This is a special case of our model with \( \delta_t(q_t) = (1 - q_t)^\gamma \) and \( u_t(q) = \beta^{t-1}(Bq)^\gamma \).

When \( u(c) = -\exp(-c) \) (a CARA case), for each \( t \), we can define \( q_t \) as the difference between \( c_t \) and a \( \frac{1}{T-t+1} \) proportion of the remaining budget, that is, \( c_1 = q_1 + \frac{B}{T}, c_2 = q_2 + \frac{B}{T} - \frac{q_1}{T_1}, c_3 = q_3 + \frac{B}{T} - \frac{q_1}{T_1} - \frac{q_2}{T-2}, \) and so on. As a result, the objective is choosing \((q_1, \ldots, q_T)\) to maximize

\[
U(q_1, \ldots, q_T) = W(c_1, \ldots, c_T) = -\exp(-q_1 - \frac{B}{T}) - \exp(-q_2 - \frac{q_1}{T} - q_1) - \ldots - \exp(-q_T - \sum_{t=1}^{T-1} \frac{q_t}{T-t}).
\]

This is a special case of our model with \( \delta_t(q_t) = \exp\left(\frac{q_t}{T-t}\right) \) and \( u_t(q) = -\beta^{t-1}\exp(-q_t - \frac{B}{T}) \).\(^{15}\)

We have thus shown that our utility setup is not only mathematically general as shown in the last section, but also applicable to a wide range of economically significant problems.\(^{16}\)

**D. Proof of proposition 3.1**

Let \( A'_t \equiv A \cup \{q^b\} \). We are to show that \([u_t(q_{tn}) + \delta_t(q_{tn})V_{t+1}(G_{A_t})] - V_{t+1}(G_{A_t})\) is no greater than \([u_t(q_{tn}) + \delta_t(q_{tn})V_{t+1}(G_{A'_t})] - V_{t+1}(G_{A'_t})\). As \( 0 < \delta_t(q_{tn}) < 1 \), it is sufficient to demonstrate \( V_{t+1}(G_{A_t}) \geq V_{t+1}(G_{A'_t}) \).

The key to the proof is to find the right backward induction hypothesis that is not too strong to render itself invalid and not too weak to help complete the inductive step. We decide on the following: for any natural number \( N \), if \( B = \{q_1, \ldots, q_N\}, C = \{q'_1, \ldots, q'_N\} \), and \( q_n \geq q'_n \) for all \( n \), then \( V_t(G_B) \geq V_t(G_C) \) for \( t = 2, \ldots, T \).

To see how the hypothesis entails \( V_{t+1}(G_{A_t}) \geq V_{t+1}(G_{A'_t}) \), we can write \( G_{A_t} = G_{A_{t-1}(s^n)} \), that is, adding a neutral signal \( s^n \geq q^b \) — the existence of

\(^{15}\)Since \( u_t \) has a negative range, \( \delta_t \) is increasing instead of decreasing.

\(^{16}\)We owe the examples to Paul Milgrom.

17
which is guaranteed by the MLRP assumption – to $A_t$ keeps the belief intact. We can apply the hypothesis with $B = A_t \cup \{s^n\}$ and $C = A'_t$.

We first prove a straightforward generalization of proposition 2 (the “only if” part) in Milgrom (1981).

Lemma D.1. Let the family of gpdf $\{f(\cdot|\theta) : \theta \in \Theta\}$ have the MLRP and a random vector $\vec{X} \equiv \{X_1, \ldots, X_N\}$ drawn independently from $f(\cdot|\theta)$ conditional on $\theta$. If $\vec{x} \geq \vec{x'}$, then $\vec{x}$ is more favorable than $\vec{x'}$, i.e., given any prior $G$, the posterior $G_{\vec{x}}$ stochastically dominates $G_{\vec{x'}}$.

Proof. By the MLRP assumption, we have for all $\theta > \tilde{\theta}$ and $n = 1, \ldots, N$, $f(x_n|\tilde{\theta})f(x'_n|\tilde{\theta}) \geq f(x_n|\theta)f(x'_n|\tilde{\theta})$. Multiplying these inequalities, we get

$$\left(\Pi_{n=1}^{N}f(x_n|\tilde{\theta})\right)\left(\Pi_{n=1}^{N}f(x'_n|\tilde{\theta})\right) \geq \left(\Pi_{n=1}^{N}f(x_n|\theta)\right)\left(\Pi_{n=1}^{N}f(x'_n|\tilde{\theta})\right).$$

But $\Pi_{n=1}^{N}f(\cdot|\theta)$ is the joint gpdf of $X$ conditional on $\theta$. We can simply apply proposition 1 in Milgrom (1981).

The base case of period $T$ is trivial, because the decision value of $q_{Tn}$ is exactly $u_T(q_{Tn})$.

Now we prove the inequality for other $t$ by backward induction. Assume that the hypothesis is true for $t+1$. By the Bellman equation, we need to show

$$\mathbb{E}_{G_B}\left[\max\{V_{t+1}(G_{B\cup A_t}), u_t(q_t) + \delta_t(q_t)V_{t+1}(G_{B\cup A_t})\}\right] \geq \mathbb{E}_{G_C}\left[\max\{V_{t+1}(G_{C\cup A'_t}), u_t(q'_t) + \delta_t(q'_t)V_{t+1}(G_{C\cup A'_t})\}\right].$$

Since $N_t$ has a fixed distribution and we can apply the law of total expectation, we only need to establish

$$\mathbb{E}_{G_B}\left[\max\{V_{t+1}(G_{B\cup A_t}), u_t(q_t) + \delta_t(q_t)V_{t+1}(G_{B\cup A_t})\}|N_t = m\right] \geq \mathbb{E}_{G_C}\left[\max\{V_{t+1}(G_{C\cup A'_t}), u_t(q'_t) + \delta_t(q'_t)V_{t+1}(G_{C\cup A'_t})\}|N_t = m\right]$$

for any natural number $m$.

We now apply a coupling technique (Ross, 1996) to construct a probability space and random sets $A_t = \{q_{t+1}, \ldots, q_{t+1m}\}$ and $A'_t = \{q'_{t+1}, \ldots, q'_{t+1m}\}$ on the space such that they follow distributions consistent with $G_B$ and $G_C$ respectively. For expository simplicity, we assume that both $G_\theta$ and $F(\cdot|\theta)$ are continuous and strictly increasing; consequently, each posterior $G$ about $\theta$ is also continuous and strictly increasing in $\theta$.
Construct a probability space with \( m + 1 \) independent random variables \( U_0, \ldots, U_m \) each uniformly distributed on \([0, 1]\) (Billingsley, 1995). Let \( \theta_B \equiv G_B^{-1}(U_0) \) and \( \theta_C \equiv G_C^{-1}(U_0) \) so that they are distributed according to \( G_B \) and \( G_C \) respectively. Let \( q_{t+1} \equiv F^{-1}(U_n|\theta_B) \) and \( q'_{t+1} \equiv F^{-1}(U_n|\theta_C) \) for each \( n = 1, \ldots, m \) so that conditional on \( U_0 \), they are distributed according to \( F(\cdot|\theta_B) \) and \( F(\cdot|\theta_C) \) respectively.

Because \( G_B \) stochastically dominates \( G_C \) by lemma D.1, we know that \( G_B \leq G_C \) and thus \( \theta_B \geq \theta_C \). Hence, \( f(\cdot|\theta_B) \) dominates \( f(\cdot|\theta_C) \) in likelihood ratio, implying \( F(\cdot|\theta_B) \leq F(\cdot|\theta_C) \) and thus \( q_{t+1} \geq q'_{t+1} \). As a result, \( G_{B\cup A_t} \) and \( G_{C\cup A_t'} \) satisfy the condition of the inductive hypothesis for \( t + 1 \). Therefore, we have \( V_{t+1}(G_{B\cup A_t}) \geq V_{t+1}(G_{C\cup A_t'}) \).

In this probability space, we have

\[
\begin{align*}
\max\{V_{t+1}(G_{B\cup A_t}), u_t(\bar{q}_t) + \delta_t(\bar{q}_t)V_{t+1}(G_{B\cup A_t})\} & \geq \max\{V_{t+1}(G_{B\cup A_t}), u_t(\bar{q}'_t) + \delta_t(\bar{q}'_t)V_{t+1}(G_{B\cup A_t})\} \\
& \geq \max\{V_{t+1}(G_{C\cup A_t'}), u_t(\bar{q}'_t) + \delta_t(\bar{q}'_t)V_{t+1}(G_{C\cup A_t'})\},
\end{align*}
\]

where the first inequality follows from assumption 1.

E. Proof of proposition 4.1

Let \( A'_t \equiv A \cup \{\bar{q}\} \). We are to show that \( [u_t(q_{tn}) + \delta_t(q_{tn})V_{t+1}(G_{A_t})] - V_{t+1}(G_{A_t}) \) is no smaller than \( [u_t(q_{tn}) + \delta_t(q_{tn})V_{t+1}(G_{A'_t})] - V_{t+1}(G_{A'_t}) \). But because \( 0 < \delta_t(q_{tn}) < 1 \), it is sufficient to demonstrate that \( V_{t+1}(G_{A_t}) \leq V_{t+1}(G_{A'_t}) \).

The inductive hypothesis is the following: for any natural number \( N \) and \( t = 2, \ldots, T \), \( V_t(G_{B^1, \ldots, B^t}) \leq V_t(G_{C^1, \ldots, C^t}) \) whenever \(|B^i| \leq |C^i| \) or \( B^t \) has no more elements than \( C^t \) for all \( i \). The desirable result \( V_{t+1}(G_{A_t}) \leq V_{t+1}(G_{A'_t}) \) is a special case with \( I = 1, B^1 = A_t, C^1 = A'_t, \) and \(|B^1| + 1 = |C^1| \).

The base case of period \( T \) is trivial.

Now we prove the inequality for other \( t \) by backward induction. Assume that the hypothesis is true for \( t + 1 \). By the Bellman equation, we need to show

\[
E_{G(B^1, \ldots, B^t)}\{\max\{V_{t+1}(G_{B^1, \ldots, B^t, A_t}), u_t(\bar{q}_t) + \delta_t(\bar{q}_t)V_{t+1}(G_{B^1, \ldots, B^t, A_t})\}\} \geq E_{G(C^1, \ldots, C^t)}\{\max\{V_{t+1}(G_{C^1, \ldots, C^t, A'_t}), u_t(\bar{q}'_t) + \delta_t(\bar{q}'_t)V_{t+1}(G_{C^1, \ldots, C^t, A'_t})\}\}.
\]

Based on how \( q_{tn} \) or \( q'_{tn} \) is drawn in this environment, we can imagine a hypothetical draw of an infinite sequence \( \{q_{tn}\}_{n=1}^\infty \), the entries of which are
independently distributed according to $h$; given $N_t$ and $N'_t$ drawn according to $G(B_1, \ldots, B_I)$ and $G(C_1, \ldots, C_I)$ respectively, $A_t$ contains the first $N_t$ entries of $\{q_{tn}\}_{i=1}^\infty$, and $A'_t$ contains the first $N'_t$. By the law of total expectation, we only need to establish

$$
\mathbb{E}_{G(B_1, \ldots, B_I)} \left[ \max \left\{ V_{t+1}(G(B_1, \ldots, B_I, A_t)), u_t(\bar{q}_t) + \delta_t(\bar{q}_t)V_{t+1}(G(B_1, \ldots, B_I, A_t)) \right\} \right] \{q_{tn}\}_{i=1}^\infty \geq \mathbb{E}_{G(C_1, \ldots, C_I)} \left[ G(C_1, \ldots, C_I, A'_t) \right] \{q_{tn}\}_{i=1}^\infty.
$$

By lemma D.1, $G(C_1, \ldots, C_I)$ stochastically dominates $G(B_1, \ldots, B_I)$. We can apply the coupling technique to construct a probability space and random variables $N_t$ and $N'_t$ which follows $G(B_1, \ldots, B_I)$ and $G(C_1, \ldots, C_I)$ respectively and satisfy $N_t \leq N'_t$. In this space, $G(B_1, \ldots, B_I, A_t)$ and $G(C_1, \ldots, C_I, A'_t)$ satisfies the premises of the inductive hypothesis, so $V_{t+1}(G(B_1, \ldots, B_I, A_t)) \leq V_{t+1}(G(C_1, \ldots, C_I, A'_t))$.

We can conclude that conditional on $\{q_{tn}\}_{i=1}^\infty$

$$
\max \left\{ V_{t+1}(G(B_1, \ldots, B_I, A_t)), u_t(\bar{q}_t) + \delta_t(\bar{q}_t)V_{t+1}(G(B_1, \ldots, B_I, A_t)) \right\} \leq \max \left\{ V_{t+1}(G(B_1, \ldots, B_I, A_t)), u_t(\bar{q}'_t) + \delta_t(\bar{q}'_t)V_{t+1}(G(B_1, \ldots, B_I, A'_t)) \right\}
$$

$$
\leq \max \left\{ V_{t+1}(G(C_1, \ldots, C_I, A'_t)), u_t(\bar{q}'_t) + \delta_t(\bar{q}'_t)V_{t+1}(G(C_1, \ldots, C_I, A'_t)) \right\},
$$

in this probability space, where $\bar{q}_t \leq \bar{q}'_t$ follows from $N_t \leq N'_t$ and how $A_t$ and $A'_t$ are constructed.