Abstract

This paper develops a theory of money and credit, and puts it to work in applications. Due to search frictions, equilibrium entails price dispersion. Agents can use cash or credit, with the former (latter) subject to inflation (transaction costs), and tend to use credit for more expensive purchases. This delivers exact solutions for money demand. An application tries to simultaneously account for observations on nominal price changes, the share of credit in micro data, and money demand in macro data. Other applications consider the impact of inflation on welfare, price dispersion, markups and participation. Also, going beyond past studies that focus on steady state, we analyze nonstationary equilibria.

JEL classification: E31, E51, E52, E42

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1 Introduction

This paper develops a theory of money and credit as competing payment instruments, then puts it to work in various applications, both qualitatively and quantitatively. We build on a framework, sometimes called the New Monetarist approach, that endeavors to take the exchange and payment processes seriously (see Section 2 for a literature review). This involves describing environments where agents trade with each other, and not only with their budget lines as in traditional general equilibrium theory, where the transactions incorporate explicit frictions. So that both cash and credit enter in a nontrivial way, we adopt the venerable idea (again see Section 2) that the former is subject to the inflation tax while the latter involves some sort of transaction costs. We consider both fixed and variable transaction costs, which turn out to work rather differently – indeed, a finding we did not anticipate is that a variable cost of credit outperforms a fixed cost in terms of theory and in terms of matching some features of the data.

A basic ingredient is search with price posting, as in Burdett and Judd (1983). This implies a price distribution \( F(p) \) such that any \( p \) in the nondegenerate support \( F \) yields the same profit: lower-price sellers earn less per unit, but make it up on the volume by selling with higher probability. We integrate this into the monetary model in Lagos and Wright (2005), with alternating centralized and decentralized markets. This is natural because at its core is an asynchronization of expenditures and receipts, which is crucial for any analysis of money or credit. In the centralized market agents consume, work, adjust money balances, settle debts and pay transaction costs. In the decentralized market they purchase other goods, not perfect substitutes for centralized market consumption, like in Burdett-Judd except with payment frictions: they can always use cash, but can also use credit if they pay a cost. Due to price dispersion, buyers sometimes make large and sometimes small payments in the decentralized market. Consistent with conventional wisdom, they tend to use credit for the former and cash for the latter.
This implies a simple demand for money that avoids an indeterminacy of equilibrium that plagues similar models, as discussed below. Moreover, it allows us to revisit classic issues about the relationship between inflation and markups, price dispersion, and welfare in a new light, driven by substitution between cash and credit. Also, the model allows us to reiterate a conceptual point about nominal stickiness. Firms here post prices in dollars, and as the money supply $M$ increases, $F(p)$ shifts so that the real distribution stays the same, but as long as the supports overlap some firms can keep the same $p$. This makes prices look sticky, even though everyone is allowed to adjust whenever they like at no cost. For a seller that sticks to his nominal price when $M$ increases, his real price falls, but the probability of a sale increases and, on net, in equilibrium, changing $p$ is simply not profitable.

While this point has previously been made by Head et al. (2012), and others mentioned below, our setup avoids a serious technical problem in that paper (again, see Section 2). Also, while Head et al. demonstrate that their model can be calibrated to quantitatively match features of price-change behavior, we go well beyond that. We show how to match salient features of the price-change data, plus observations on the fractions of cash and credit in the micro payments data, and money demand (the relation between real balances and interest rates) in the macro data. Head et al. have no credit, and so obviously cannot match the shares in the micro data, and do not match the macro money demand observations very well, either. Versions of the model presented below do a reasonable job matching these data simultaneously, which is desirable because they all concern monetary phenomena and all have implications for monetary policy.¹

¹To be clear, our theory does not put tight predictions on individual repricing – that is indeterminate, although this is not payoff relevant while the indeterminacy associated with the problem in Head et al. (2012) is payoff relevant. Also, our firms sometimes use tie-breaking rules to decide whether to change $p$ when they are indifferent, but this is nothing like Calvo models, where they sometimes cannot change $p$ even when desperate to do so. In any case, the point of this application is that there are equilibria consistent with repricing data even without Calvo arrivals, menu costs or irrational inattention. This does not prove those ideas are unimportant, of course, only that they are not necessary to explain certain observations.
In addition to revisiting nominal rigidities, we uncover some novel effects of inflation in money-credit economies. First, the welfare impact of inflation is small: with a variable cost, e.g., eliminating inflation of \( \pi = 0.10 \) is worth only 0.2\% of consumption. This is because, in our baseline setting, \( \pi \) does not affect the intensive or extensive margin of decentralized trade (the size or number of trades), and welfare costs come only from impinging on cash-credit margins. While other studies find other channels via which \( \pi \) matters, our goal is to highlight a different mechanism. Additionally, we show how \( F(p) \) varies with \( \pi \), and how both markups and price dispersion decrease, consistent with some empirical findings.

We also present an extension that introduces endogenous participation, which can change the results, since then policy does affect the extensive margin of trade. And we present an extension that goes beyond the steady state analyses by explicitly analyzing dynamic equilibria. This allows us to demonstrate how inflation affects the economy as it evolves over time, for different reasons, not only how it responds to perfectly anticipated monetary injections.

The specification with a fixed cost of credit delivers a closed-form solution for money demand, reminiscent of Baumol-Tobin, and with a similar interpretation, although arguably our version has some advantages (it emerges in a general equilibrium context where money and credit are essential and their roles are explicit). This model can match money demand observations, but not the shares of money and credit in the micro data, given our calibration procedure. The specification with a proportional cost of credit also delivers a nice money demand function, if slightly more complicated, and can simultaneously match money demand, the payment data, and the key price-change facts, including long average durations, large average changes, many small and negative changes, a decreasing hazard, and repricing behavior that varies with inflation. Moreover, a variable cost avoids non-convexities, and is in other ways more tractable, too. Hence, we conclude that the variable- outperforms the fixed-cost model on several dimensions.
The rest of the presentation is organized as follows. Section 2 reviews the literature. Section 3 describes the basic model. Sections 4 and 5 present stationary equilibrium analysis and calibration. Section 6, 7 and 8 consider applications to price-change behavior, the cost of inflation, and endogenous participation. Section 9 studies dynamics. Section 10 concludes.

2 Literature

Here we discuss related work in several areas (although one can skip this for now and jump to the model in Section 3 without loss of continuity). We do not survey the New Monetarist literature, as that was recently done in Lagos et al. (2015), but want to mention previous efforts to integrate Burdett-Judd pricing into these kinds of models. Head et al. (2012) and Wang (2014) embed Burdett-Judd in Lagos and Wright (2005), while Head and Kumar (2005) and Head et al. (2010) embed it in Shi (1997). There is, however, a technical problem with using the baseline Burdett-Judd formulation, with indivisible goods and price posting, in a monetary model: It leads to an indeterminacy (i.e., a continuum) of stationary equilibria.\(^2\)

The above-mentioned papers get around this by assuming divisible goods, but then another problem pops up: What should firms post? They assume linear menus, where sellers set \(p\) and let buyers choose any \(q\) as long as they pay \(pq\), but there is no reason to think this maximizes profit.

In the environment with costly credit, we eliminate this indeterminacy in a different way while maintaining indivisible goods, and thereby avoid ad hoc assumptions like linear pricing. Intuitively, holding more cash reduces the amount of costly credit that buyers expect to use, which delivers a well-behaved money demand function and a unique monetary equilibrium. Note that we do not take a

\(^2\)This comes up in a series of papers spawned by Green and Zhou (1998). See Jean et al. (2010) for citations and further discussion, but here is a simple versions of the idea: If all sellers post \(p\) then buyers’ best response is to bring \(m = p\) dollars to the market as long as \(p\) is not too big. If all buyers bring \(m\) then sellers’ best response is post \(p = m\) as long as \(m\) is not too small. Hence, all \(p = m\) in some range constitute equilibria, and they are not payoff equivalent.
stand on whether divisible or indivisible goods are more realistic, as that depends on the context, but clearly indivisibility is an assumption on the physical environment, which seems better than a restriction on pricing strategies. Also note that the indeterminacy in question concerns stationary equilibria, not dynamic equilibria, which are indeterminate in most models of money and credit, including this one, as we show below by way of contact with Kiyotaki and Moore (1997), Gu et al. (2013) and references therein.

Despite technical differences, we share with Head et al. (2010) the goal of analyzing price-change data without imposing menu costs (e.g., Mankiw 1985), forcing sellers to change only at exogenous points in time (e.g., Taylor 1980; Calvo 1983), or assuming rational inattention (e.g., Woodford 2002; Sims 2003). While those approaches are obviously interesting, we focus on search to see how far that can go alone.\textsuperscript{3} Seminal work by Caplin and Spulber (1987) and Eden (1994) takes a similar approach to sticky prices, but does not confront the data as we do, and does not use the same microfoundations, which we think are worth considering since these models have proved fruitful in other contexts. Within the set of papers that use similar theory, when they do confront the data, we go further by trying to simultaneously account for price-change behavior, payment methods in the micro data, and money demand in the macro data, which are natural to study jointly because they all concern monetary phenomena and monetary policy.

There is much empirical work on price adjustment. Campbell and Eden (2014) find in grocery-store data an average duration between price changes of around 10 weeks, but we do not want to focus exclusively on groceries. Bils and Klenow (2004) find in BLS data at least half of prices last less than 4.3 months, or 5.5 months if one excludes sales. Klenow and Kryvtsov (2008) report durations from 6.8 to 10.4 months, while Nakamura and Steinsson (2008) report 8 to 11 months, excluding

\textsuperscript{3}Burdett and Menzio (2014), e.g., combine search and menu costs, which makes the analysis much more difficult even without money. Other nonmonetary search models with menu costs include Benabou (1988, 1992a) and Diamond (1993).
substitutions and sales. These papers also find large fractions of small and negative price changes, plus some evidence of a decreasing hazard. Eichenbaum et al. (2011) report a duration of 6 months for reference prices (those most often quoted in a quarter). Cecchetti (1986) finds durations for magazine prices from 1.8 months to 14 years, while Carlton (1986) finds durations for wholesale prices from 5.9 months for household appliances to 19.2 for chemicals. More research along these lines is surveyed by Klenow and Malin (2010).

We try to take this empirical work into account to impose discipline on the theory. One issue emphasized in the literature, e.g., is that average price changes are fairly big, suggesting high menu costs, but there are also many small changes, suggesting low menu costs. Midrigan (2011) explains this by having firms sell multiple products, and paying a cost to change one price lets them change the rest for free. See also Vavra (2014). We like these approaches, but are still interested in alternatives. Our model accounts for realistic durations, large average changes, many small and negative changes, and repricing behavior that depends on inflation. It can also yield a decreasing hazard, which is problematic for some other approaches (Nakamura and Steinsson 2008), and generates price dispersion at low or zero inflation, consistent with evidence but not some other models (Campbell and Eden 2014). While every approach has advantages and disadvantages, these findings suggest that search-based theories should be part of the conversation on price stickiness.\(^4\)

On the cost of inflation, see Lucas (2000) or Cooley (1995, Chapter 7) for discussions using money-in-the-utility-function or cash-in-advance specifications, where typically eliminating an annual inflation of \(\pi = 0.10\) is worth around 0.5% of

\(^4\)In discussions with people in the area, we found there is more or less agreement that these are the stylized facts: (1) Prices change slowly, but exact durations vary across studies. (2) The frequency and size of changes vary across goods. (3) Two sellers changing at the same time usually do not pick the same \(\bar{p}\). (4) Many changes are negative. (5) Hazards decline slightly with duration. (6) There are many small (below 5\%) and many big (above 20\%) changes. (7) The frequency and size of changes, and fraction of negative changes, vary with inflation. (8) There is price dispersion even at low inflation. Our models are consistent with all these facts.
consumption.\textsuperscript{5} Lagos et al. (2015) surveys search-and-bargaining models that get numbers closer to 5.0\%. Again, our benchmark findings are much smaller. On inflation and price dispersion, empirical findings are mixed – e.g., Parsley (1996) and Debelle and Lamont (1997) find a positive relation, Reinsdorf (1994) finds a negative relation, and Caglayan et al. (2008) find a U-shaped relation. On markups and inflation, Benabou (1992b) reports a small but significant negative relationship. Benabou (1992a) and Head and Kumar (2005) explain this by inflation increasing dispersion and hence search effort. Inflation decreases dispersion and markups here instead by directly affecting the endogenous payment methods.

As regards money demand, we deliver exact solutions reminiscent of well-known results by Baumol (1952), Tobin (1956), Miller and Orr (1966) and Whalen (1966). The economic intuition is similar, involving a comparison between the opportunity costs of holding cash and tapping financial services. However, those papers involve partial-equilibrium analyses, or more accurately, decision-theoretic analyses of how to manage one’s money given that it must be used for payments. While such models are still being used to good effect (e.g., Alvarez and Lippi 2009, 2014), and we recognize this is partly a matter of taste, we like our structure because it is tractable and easy to integrate with standard equilibrium macro. Incorporating price posting and costly credit is also a further step in understanding the New Monetarist framework, which has recently become a workhorse in the area.\textsuperscript{6}

Modeling money and credit jointly is a long-standing challenge for theory, and generally recognized as potentially very relevant for policy. One approach, including a large number of papers following Lucas and Stokey (1987), simply assumes that some goods require cash while other allow credit. Closer to us, there is a long

\textsuperscript{5}We cannot review here all these papers in detail, but mention Dotsey and Ireland (1996) and Aiyagari et al. (1998) as particularly related to us in their concern for transaction costs

\textsuperscript{6}This is not to suggest that all monetary theorists buy into the approach – e.g., see Wallace (2014). In any case, previous analyses of these models use Nash, Kalai or strategic bargaining, directed search, competitive price taking, auctions and pure mechanism design. See Lagos et al. (2015) for citations, plus applications in finance, labor, international, growth etc.
tradition of allowing individuals to choose cash or credit after incorporating a cost to using the latter, including Prescott (1987), Freeman and Huffman (1991), Chatterjee and Corbae (1992), Lacker and Schreft (1996) and Freeman and Kydland (2000). See Nosal and Rocheteau (2011, Chapter 8) for a general discussion, and Gomis-Porqueras and Sanches (2013), Li and Li (2013), Bethune et al. (2015) and Lotz and Zhang (2015) for more recent work that is related in spirit to our approach. Various interpretations for transaction costs are suggested in those papers. We are agnostic, and willing to entertain the resources used up in record keeping, screening, enforcement etc. It is also possible, as in Gomis-Porqueras et al. (2014), e.g., to interpret the cost as a tax, where buyers using credit in are subject to a levy that can be avoided by paying in cash. Or, as emphasized in Gu et al. (2015) and references therein, the cost can described in terms of monitoring.\footnote{As Wallace (2013) says, “If we want both monetary trade and credit in the same model, we need something between perfect monitoring and no monitoring. As in other areas of economics ... extreme versions are both easy to describe and easy to analyze. The challenge is to specify and analyze intermediate situations.” Here monitoring is available but not free – which is not especially ‘deep,’ but is useful in applications. We can suggest Araujo and Hu (2014) as an attempt at a ‘deeper’ approach, based on mechanism design.}

Finally, we mention heterogeneity. As is well known in other applications of Burdett-Judd, including the large labor literature following Burdett and Mortensen (1998), if firms are homogeneous then theory does not pin down which one charges which \( p \), only the distribution \( F(p) \). With heterogeneity, however, lower-cost firms prefer lower \( p \) since they like higher volume. Still, for any subset of sellers with the same marginal cost, it does not matter which one posts which \( p \). This is especially relevant for retail, where marginal cost is the wholesale price. Even if a few retailers get better deals (e.g., quantity discounts), many face the same wholesale terms. Irrespective of fixed costs, wages etc., these sellers are homogeneous for our purposes, and there is indeterminacy with respect to which one sets which price. We mention this because it bears on one of our applications, the discussion of sticky prices, even if it is unimportant for all the others.
3 Environment

Each period in discrete time has two subperiods: first there is a decentralized market, called BJ for Burdett-Judd; then there is a frictionless centralized market, called AD for Arrow-Debreu. There is a set of firms (retailers) with measure 1, and a set of households with measure $\bar{b}$. Households consume a divisible good $x_t$ and supply labor $\ell_t$ in AD, while in BJ they consume an indivisible good $y_t$ produced by firms at unit cost $\gamma \geq 0$. As agents are anonymous in the BJ market, they cannot use credit unless they access a technology to authenticate identity and record transactions at a cost. By incurring the cost, they can get BJ goods in exchange for commitments to deliver $d_t$ dollars in the next AD; otherwise they need cash at point of sale. We consider both a fixed cost $\delta$ and a proportional cost $\tau$. The transaction cost in general is $C(d_t) = \delta \mathbf{1}(d_t) + \tau d_t$, where $\mathbf{1}(d_t)$ is an indicator function that is 1 iff $d_t > 0$. Here the cost is paid by households, but the outcome is identical if it is paid by firms, as in elementary tax-incidence theory.

Household utility within a period is $U(x_t) + \mu \mathbf{1}(y_t) - \ell_t$, where $U''(x_t) > 0 > U'''(x_t)$, $\mu > \gamma + \delta$ and $\mathbf{1}(y_t)$ is an indicator function. Let $\beta = 1/(1+r)$, with $r > 0$, be a discount factor between AD today and BJ tomorrow; any discounting between BJ and AD can be subsumed in the notation. We impose $\pi > \beta - 1$, where in stationary equilibrium $\pi$ is the inflation rate, and the nominal interest rate is given by the Fisher equation $1 + i = (1 + \pi)(1 + r)$. Note that $\pi > \beta - 1$ implies $i > 0$, and the Friedman rule is the limiting case $i \to 0$. As usual, $1 + i$ is the amount of money agents require in the next AD market to give up a dollar in the current AD market, and whether or not such trades occur in equilibrium we can price them. Let $x_t$ be AD numeraire, and assume it is produced one-for-one with $\ell_t$, so the real wage is 1. The AD price of money in numeraire is $\phi_t$, and $1/\phi_t$ is the nominal price level.

Agents enter the BJ market for free for now (later we introduce an entry cost). Each firm in BJ maximizes profit by posting a price, taking as given the CDF
of other firms’ prices, $F_t(p)$, with support $\mathcal{F}_t$. Every period a household in BJ randomly samples $n$ firms – i.e., sees $n$ independent draws from $F_t(p)$ – with probability $\alpha_n$. For our purposes it suffices to have $\alpha_1, \alpha_2 > 0$ and $\alpha_n = 0 \forall n \geq 3$, but this can be generalized in various ways: e.g., Burdett et al. (2014) study different parametric specifications, while the original Burdett and Judd (1983) paper shows how to endogenize $\alpha_n$, with $\alpha_n = 0 \forall n \geq 3$ as an outcome, not an assumption (this works here, too, but we prefer to avoid the increased notation).

The money supply per buyer evolves according to $M_{t+1} = (1 + \pi) M_t$, with changes implemented in AD via lump-sum taxes or transfers, although the relevant results are the same if instead the government uses seigniorage to buy AD goods.

### 3.1 Firm Problem

Expected real profit for a firm posting $p$ at date $t$ is

$$\Pi_t(p) = b_t \left[ \alpha_1 + 2\alpha_2 \hat{F}_t(p_t) \right] (p\phi_t - \gamma),$$

where $\hat{F}_t(p) \equiv 1 - F_t(p)$. Thus, net revenue per unit is $p\phi_t - \gamma$, and the number of units is determined as follows: The probability a household contacts this firm and no other is $\alpha_1$. Then the firm makes a sale for sure. The probability a household contacts this firm plus another is $2\alpha_2$, as it can happen in two ways, this one first and the other one second, or vice versa. Then the firm makes a sale iff it beats the other firm’s price, which happens with probability $\hat{F}_t(p)$. This is all multiplied by tightness $b_t$ to convert buyer probabilities into seller probabilities.

Profit maximization means every $p \in \mathcal{F}_t$ yields the same profit. As is standard in these models, $F_t(p)$ is continuous and $\mathcal{F}_t = [p_t, \bar{p}_t]$ is an interval.\(^8\) Taking as given for now $\bar{p}_t$, and anticipating that $\bar{p}_t$ is not so high that buyers reject it,\(^8\) there cannot be a mass of firms with the same $p$ because any one of them would have a profitable deviation to $p - \varepsilon$, since they lose only $\varepsilon$ per unit and make discretely more sales by undercutting others at $p$. Also, if there were a gap between $p_1$ and $p_2 > p_1$, a firm posting $p_1$ can deviate to $p_1 + \varepsilon$ and earn more per unit without losing sales.
\[ \forall p \in \mathcal{F}_t \text{ profit from } p \text{ must equal profit from } \bar{p}_t, \text{ which is} \]

\[ \Pi_t(\bar{p}_t) = b_t \alpha_1 (\bar{p}_t \phi_t - \gamma), \quad (2) \]

because the highest price firm never beats the competition. Equating (1) to (2) and rearranging immediately yields the form of equilibrium distribution:

**Lemma 1** \[ \forall p \in \mathcal{F}_t = [\underline{p}_t, \bar{p}_t] \]

\[ F_t(p) = 1 - \frac{\alpha_1}{2 \alpha_2} \frac{\phi_t \bar{p}_t - \phi_t p}{\phi_t p - \gamma}. \quad (3) \]

It is easy to check \( F_t'(p) > 0 \) and \( F_t''(p) < 0 \). Also, using \( F(\bar{p}_t) = 0 \) we get

\[ p_t = \frac{\alpha_1 \phi_t \bar{p}_t + 2 \alpha_2 \gamma}{\phi_t (\alpha_1 + 2 \alpha_2)}. \quad (4) \]

To translate from dollars to numeraire, let \( q_t = \phi_t p_t \) and write the real price distribution of as

\[ G_t(q) = 1 - \frac{\alpha_1}{2 \alpha_2} \frac{\tilde{q}_t - q}{\tilde{q}_t - \gamma}. \quad (5) \]

We denote its support by \( \mathcal{G}_t = [\underline{q}_t, \bar{q}_t] \), and let \( \hat{G}_t(q_t) \equiv 1 - G_t(q_t) \).

### 3.2 Household Problem

To facilitate the presentation, consider for now a stationary equilibrium, where real variables are constant and nominal variables grow at rate \( \pi \). Framing the household problem in real terms, the state variable in AD is net worth, \( A = \phi m - \phi d - C(d) + I \), where \( \phi m \) and \( \phi d \) are real money balances and debt carried over from the previous BJ market, \( C(d) \) is the transaction cost of having used credit, and \( I \) is any other income. Generally, \( I \) includes transfers net of taxes, plus profit, since as in standard general equilibrium theory, the firms here are are owned by the households, which plays no role except for making the welfare criterion unambiguous. All debt is settled in AD, so that households start BJ with a clean slate – i.e., they could roll over \( d \) from one AD market to the next, but since preferences are linear in \( \ell \), there is no point. Hence the state variable in BJ is real balances, \( z \).
The AD and BJ value functions are $W(A)$ and $V(z)$. These satisfy

$$W(A) = \max_{x,\ell,z} \{ U(x) - \ell + \beta V(z) \} \text{ st } x = A + \ell - (1 + \pi) z,$$

where the cost of real balances $z$ next period is $(1 + \pi) z$ in terms of numeraire this period. Eliminating $\ell$ and letting $x^*$ solve $U'(x^*) = 1$, after rearranging, we get

$$W(A) = A + U(x^*) - x^* + \beta \max_z O_i(z)$$

where the objective function for the choice of $z$ is $O_i(z) \equiv V(z) - (1 + i) z$, with $i$ given by the Fisher equation. Then this result is immediate:

**Lemma 2** $W'(A) = 1$ and the choice of $z$ does not depend on $A$.

The BJ value function satisfies (see the Appendix for more detail)

$$V(z) = W(z + I) + (\alpha_1 + \alpha_2) \left[ \mu - \mathbb{E}_H q - \delta \hat{H}(z) - \tau \mathbb{E}_H \max(0, q - z) \right],$$

where $\hat{H}(q) \equiv 1 - H(q)$, and $H(q)$ is the CDF of transaction prices,

$$H(q) = \frac{\alpha_1 G(q) + \alpha_2 \left[ 1 - \hat{G}(q)^2 \right]}{\alpha_1 + \alpha_2}.$$

Notice $H(q)$ differs from the CDF of posted prices $G(q)$, because a buyer seeing multiple draws of $q$ obviously picks the lowest. Also, notice the costs $\delta$ and $\tau (q - z)$ are paid iff $q > z$. Therefore, in terms of simple economics, the benefit of higher $z$ is that it reduces the expected cost of having to tap credit.

## 4 Equilibrium

The above discussion characterizes behavior given $\bar{q}$, which will be determined presently. First we have these definitions:

**Definition 1** A stationary equilibrium is a list $(G(q), z)$ such that: given $G(q)$, $z$ solves the household’s problem; and given $z$, $G(q)$ solves the firms’ problem with $\bar{q}$ determined as in Lemma 3 below.
Definition 2 A nonmonetary equilibrium, or NME, has \( z = 0 \), so all BJ trades use credit. A mixed monetary equilibrium, or MME, has \( 0 < z < \bar{q} \), so BJ trades use cash for \( q \leq z \) and credit for \( q > z \). A pure monetary equilibrium, or PME, has \( z \geq \bar{q} \), so all BJ trades use cash.

Other variables, like \( x \) and \( \ell \), can be computed, but are not needed in what follows. Also, notice in NME prices must be described in numeraire \( q \), while in MME or PME they can equivalently be described in numeraire or dollars.

The next step is to describe \( \bar{q} \). The following results are proved in the Appendix except where obvious:

Lemma 3 IN NME, \( z = 0 \) and \( \bar{q} = (\mu - \delta) / (1 + \tau) \). In MME, \( z \in (0, \mu - \delta) \) and \( \bar{q} = (\mu - \delta + \tau z) / (1 + \tau) \). In PME, \( \bar{q} = z \geq \mu - \delta \).

Lemma 4 In MME, \( O_i(z) \) is continuous. It is smooth and strictly concave \( \forall z \in (\underline{q}, \bar{q}) \), and linear \( \forall z \notin (\underline{q}, \bar{q}) \).

4.1 Fixed Cost

Consider first \( \tau = 0 < \delta \). Given \( \delta < \mu - \gamma \), there is a nonmonetary equilibrium where all BJ transactions use credit – basically, the original Burdett-Judd equilibrium. We are more interested in monetary equilibrium. As shown in Figure 1, as a special case of Lemma 4, the objective function \( O_i(z) \) is linear \( \forall z \notin (\underline{q}, \bar{q}) \) with slope \( O_i'(z) = -i < 0 \), since real balances only reduce the expected cost of credit when \( z \in G_i \). It is also easy to check \( O_i''(z) < 0 \ \forall z \in (\underline{q}, \bar{q}) \).

These results imply \( \exists! z_i = \arg \max_{z \in [\underline{q}, \bar{q}]} O_i(z) \). If \( z_i \in (\underline{q}, \bar{q}) \), as required for MME, it satisfies the FOC

\[
(\alpha_1 + \alpha_2) \delta H'(z_i) = i. \tag{10}
\]

To check \( z_i \in (\underline{q}, \bar{q}) \), let \( \hat{z}_i \) be the global maximizer of \( O_i(z) \), and let \( O_i^- (z) \) and \( O_i^+ (z) \) be the left and right derivatives. If \( O_i^+(\underline{q}) \leq 0 \) then \( \hat{z}_i = 0 \), as in the left
Figure 1: Possible Equilibria with Fixed Cost

panel of Figure 1. If \( O_i^+(q) > 0 \) then we need to check \( O_i^-(\bar{q}) \). If \( O_i^-(\bar{q}) \geq 0 \) then either \( \hat{z}_i = 0 \) or \( \hat{z}_i = \bar{q} \), as in the center panel. If \( O_i^-(\bar{q}) < 0 \) then either \( \hat{z}_i = 0 \) or \( \hat{z}_i = z_i \), as in the right panel. As a result (see the Appendix for details), we have:

**Proposition 1** In the fixed-cost model:

(i) \( \exists! \) a unique NME;

(ii) \( \exists! \) MME iff \( \delta < \bar{\delta} \) and \( i \in (\hat{i}, \bar{i}) \);

(iii) \( \exists \) PME iff either \( \bar{\delta} < \delta < \mu - \gamma \) and \( i < \hat{i} \), or \( \delta < \bar{\delta} \) and \( i < \hat{i} \);

where the thresholds satisfy \( \bar{i} \in (\hat{i}, \infty) \),

\[
\hat{i} = \frac{\delta \alpha_1^2}{2 \alpha_2 (\mu - \delta - \gamma)} \quad \text{and} \quad \bar{\delta} = \mu - \frac{\gamma (2 \alpha_2^2 + 2 \alpha_1 \alpha_2)}{2 \alpha_2^2 + 2 \alpha_1 \alpha_2 - \alpha_1^2}.
\]

This is illustrated in Figure 2, where notice money (credit) may be used iff the nominal rate \( i \) (transaction cost \( \delta \)) is not too high.\(^9\) We are mainly interested in MME. When it exists, it is easy to insert \( G(q) \) into (10) and rearrange to get explicit the solution for

\[
\hat{z}_i = \gamma + \left[ \alpha_1^2 \delta (\mu - \delta - \gamma)^2 / 2 \alpha_2 \right]^{1/3} i^{-1/3}.
\]  

\(^9\)Notice also that there is a continuum of PME when they exist, for the reasons discussed in fn.2. For the other cases we have uniqueness – e.g., while NME and MME may coexist, there is at most one of each.
Figure 2: Equilibria in Parameter Space with Fixed Cost

The money demand function – i.e., the solution for real balances as a function of the nominal rate $i$ – in (11) is reminiscent of the famous square-root rule in Baumol (1952) and Tobin (1956), or cube-root rule in Miller and Orr (1966) and Whalen (1966). In those models, the usual story has an agent sequentially incurring expenses requiring currency, with a fixed cost of rebalancing. The decision rule compares $i$, the opportunity cost of cash, with the benefit of reducing the number of financial transactions interpretable as trips to the bank. Our buyers make at most one transaction in before rebalancing $z$, but its size is random. Still, they compare $i$ with the benefit of reducing the use of financial services, again interpretable as trips to the bank, although one might say they now go there to get a loan and not to make a withdrawal.\footnote{See Berentsen et al. (2007) or Chiu and Meh (2011) on explicitly adding banking in a way consistent with our approach, but we are really using trips to the bank only heuristically, as in the standard textbook discussion of Baumol-Tobin.}

4.2 Variable Cost

Now consider $\tau > 0 = \delta$. This setup is in many respects easier, and avoids a technical issue with fixed costs that we waited until now to raise: In economies with nonconvexities like fixed costs, it can be desirable to let agents trade using
lotteries. One might try to argue that lotteries are infeasible or unrealistic, but that would be awkward, because ruling them out is uncomfortably close to ruling out nonlinear menus, something we criticized above. We do not analyze lotteries because the variable-cost model actually works better and it has no role for lotteries. We still covered the fixed-cost case since it has a tradition, and we wanted to compare the two environments.

The price distribution emerging from the firms’ problem is similar to the fixed-cost model, and in particular,

\[
\bar{q} = \frac{\mu + z\tau}{1 + \tau} \quad \text{and} \quad q = \frac{\alpha_1(\mu + z\tau) + 2\alpha_2\gamma(1 + \tau)}{(\alpha_1 + 2\alpha_2)(1 + \tau)}.
\]

It is easy to check that \(O_i(z)\) is now differentiable everywhere, including \(q = \bar{q}\) and \(q = \bar{q}\). As Figure 3 shows, this means there are only two possible types of equilibria: if \(i > (\alpha_1 + \alpha_2)\tau\) there is a unique NME; if \(i < (\alpha_1 + \alpha_2)\tau\) there is a unique MME; there is no PME. Letting \(i^* = i^*(\tau)\) be the \(i\) that drives buyers’ payoff to 0, we have this:

\[11\]

See Berentsen et al. (2002) for an analysis in related monetary models. The idea here would be for a seller to post: “you get my good for sure if you pay \(p\); if you pay \(p < \tilde{p}\) then you get my good with probability \(P = P(\tilde{p})\).” In Section 4.1, when a buyer with \(m = p - \varepsilon\) meets a seller posting \(p\), he pays \(p - \varepsilon\) in cash, \(\varepsilon\) in credit and \(\delta\) in fixed costs; if \(\varepsilon\) is small, both parties would prefer to trade using cash only, to avoid \(\delta\), and have the good delivered with probability \(P < 1\).
Figure 4: Equilibria in Parameter Space with Variable Cost

**Proposition 2** In the variable-cost model

(i) $\exists!$ a unique NME iff $\tau \leq \mu / \gamma - 1$;

(ii) $\exists!$ a unique MME iff $i < \min \{ \tau (\alpha_1 + \alpha_2), i^* \}$;

(iii) $\nexists$ a PME.

As Figure 4 illustrates, now MME exists $\forall \tau$ as long as $i$ is not too big. From (10) we get the money demand function

$$\hat{z}_i = \gamma + \frac{(\mu - \gamma) \left( \tau + (1 + \tau) \sqrt{1 + 4\alpha_2 i / \alpha_1^2 \tau} \right)}{1 + 2\tau + 4\alpha_2 (1 + \tau)^2 i / \alpha_1^2 \tau}. \quad (12)$$

While this is somewhat different from the fixed-cost model, as shown below both do a reasonable job of fitting the money demand data. However, the variable-cost model can match some other facts better.

### 4.3 Repricing Behavior

While this is not the first paper to make the point, and this is not the main point of the paper, here we sketch the argument behind sticky prices. With either fixed or variable costs, the nominal price distribution $F_t(p)$ is uniquely determined, but an individual firm’s price is not. Figure 5 is drawn for the calibrated parameters in Section 5.2. With $\pi > 0$, the density $F'_{t+1}$ lies to the right of $F'_t$. Firms with
$p < p_{t+1}$ at $t$ (Region A) must reprice at $t + 1$, because while $p$ maximized profit at $t$, it no longer does so at $t + 1$. But as long as the supports $\mathcal{F}_t$ and at $\mathcal{F}_{t+1}$ overlap, there are firms with $p > p_{t+1}$ at $t$ (Region B) that can keep the same $p$ at $t + 1$ without reducing profit. They are allowed to change, at no cost, but it is simply not profitable.

![Figure 5: Nominal Price Densities and Inflation](image)

Given this, consider a particular repricing strategy. If $p_t \notin \mathcal{F}_{t+1}$ then $p_{t+1}(p_t) = \hat{p}$ where $\hat{p}$ is a new price; and if $p_t \in \mathcal{F}_{t+1}$ then

$$p_{t+1}(p_t) = \begin{cases} 
  p_t & \text{with prob } \sigma \\
  \hat{p} & \text{with prob } 1 - \sigma 
\end{cases}.$$  

(13)

Notice $\sigma$ defines a payoff-irrelevant tie-breaking rule. This is very different from Calvo pricing, where firms desperate to change $p$ are simply not allowed. Here it is firms that are indifferent that stick to their current $p$ with probability $\sigma$. Then consider symmetric equilibrium, where all changers pick a new $\hat{p}$ from the same repricing distribution $R_{t+1}(\hat{p})$. As in Head et al. (2012), given $\sigma$, there is a unique such distribution that generates the equilibrium $F_{t+1}(p)$:

$$R_{t+1}(p) = \begin{cases} 
  F_t\left(\frac{p}{\hat{p}}\right)^{-\sigma[F_t(p) - F_t(p_{t+1})]} & \text{if } p \in [p_{t+1}, \hat{p}_t) \\
  \frac{F_t\left(\frac{\hat{p}}{p}\right)^{-\sigma[F_t(p_{t+1}) - F_t(p)]}}{1-\sigma+\sigma F_t(p_{t+1})} & \text{if } p \in [\hat{p}_t, \hat{p}_{t+1}] 
\end{cases}.$$ 

(14)
It is now routine to compute statistics from the model and compare these to the facts that are deemed interesting in the literature. In the present exercise the concentration is on the distribution of price changes, \((p_{t+1} - p_t)/p_t\), because that is the topic of discussion in the research to which we are trying to contribute. Notice that different values of \(\sigma\) imply different repricing behavior – but that does not at all mean anything goes. We use data below to pin down \(\sigma\), and given \(\sigma\) there is a unique symmetric repricing distribution. We then ask how the model does on other dimensions. While we are well aware that the theory does not impose tight restrictions (recall fn. 1), the model is nonetheless useful at least for demonstrating that the informational content of price-change data may be different from what is suggested by Mankiw-style costs or Calvo-style arrivals, and we suggest that this is a point worth re-emphasizing.

5 Quantitative Results

We want to match the fractions of money and credit usage in payment data, plus statistics derived from a standard empirical notion of money demand. Following Lucas (2000), this notion is \(L_i = \hat{z}_i/Y\), where \(Y = x^* + \alpha_1 + \alpha_2\) \(E_H q\) is output aggregated over AD and BJ. If \(U(x) = \log(x)\) then \(x^* = 1\). Explicit formulae for \(L_i\) and its elasticity \(\eta_i\) are given in the Appendix, and we target these in the data. Another key statistic is the average BJ markup \(E_G q/\gamma\), computed using posted prices. This is key because BJ equilibrium can deliver anything from monopoly to marginal-cost pricing as \(\alpha_1/\alpha_2\) varies, and so \(E_G q/\gamma\) contains information about the \(\alpha\)'s. We also use the average duration between changes, which pins down \(\sigma\) in the tie-breaking rule. Finally, we target the average absolute price change.

\[\text{This is obviously a normalization. Also, if we write utility as } \log(x) + \mu 1(y) - \psi \ell, \text{ with } \mu \text{ capturing the relative importance of BJ vs AD goods and } \psi \text{ the importance of leisure, as in standard business cycle research, } \psi \text{ can be set to match average hours. But as the results below are independent of hours, we ignore this.}\]
5.1 Data

We focus on 1988-2004, because the price-change observations are from that period, although in principle information from other periods can also be used to calibrate parameters. For money, the best available data is the M1J series in Lucas and Nicolini (2012) that adjusts M1 for money-market deposit accounts, similar to the way M1S adjusts for sweeps as discussed in Cynamon et al. (2006). Lucas-Nicolini provide an annual series from 1915-2008 and a quarterly series from 1984-2013, and make the case that there is a stable relationship between these and (3-month T-Bill) nominal interest rates. We use their quarterly series, because the years correspond better to the price-change sample. In these data the average annualized nominal rate is $\bar{E}_i = 0.048$, which implies $L_{\bar{E}_i} = 0.279$ and $\eta_{\bar{E}_i} = -0.149$.

Markup information comes from the U.S. Census Bureau Annual Retail Trade Report 1992-2008. At the low end, in Warehouse Clubs, Superstores, Automotive Dealers and Gas Stations, gross margins over sales range between 1.17 and 1.21; at the high end, in Specialty Foods, Clothing, Footwear and Furniture, they range between 1.42 and 1.44. Our target for the gross margin is 1.3, in the middle of these numbers. This implies a markup of 1.39, as in Bethune et al. (2014). While this number is above what macro people often use, it is very much consistent with the micro data analyzed by Stroebel and Vavra (2015). Moreover, the exact value does not matter much over a reasonable range, similar to the findings in Aruoba et al. (2011).

On the fraction of money and credit transactions, there are various sources. First, in terms of concept, we follow the related literature by interpreting monetary transactions broadly to include cash, check and debit card purchases. As a rationale, we offer three points: (1) Checks and debit cards use demand deposits, which like currency are liquid and pay basically no interest, and it is irrelevant

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13In the longer annual sample, $\bar{E}_i = 0.038$, $L_{\bar{E}_i} = 0.257$ and $\eta_{\bar{E}_i} = -0.105$. Using these instead does not affect the results much. We also tried truncating the data in 2004, to better match the pricing sample and eliminate the recent crisis; which also did not affect the results very much.
for our theory whether one’s money is in one’s pocket or checking account. (2) A key feature of credit is that it allows buyers to pay for BJ goods by working in the next AD market, while cash, check and debit purchases all require working in the previous AD market, and this matters especially because BJ transactions are random. (3) This notion of money in the micro data is consistent with the use of M1J in the macro data.

Older calibrations of monetary models (Cooley 1995) target 16% for credit purchases, but more information is now available. In detailed grocery-store data from 2001, Klee (2008) finds credit cards account for 12% of purchases, but we do not want focus too much on just groceries. In 2012 Boston Fed data, as discussed by Bennett et al. (2014) and Schuh and Stavins (2014), credit cards account for 22% of purchases in their survey and 17% in their diary sample. In Bank of Canada data, as discussed by Arango and Welte (2012), the number is 19%. While not literally identical, the Boston Fed and Bank of Canada data are close, and suggest a target of 20%. Note that this number does not change too much over time, where the bigger evolution has been into debit and out of checks, and to some extent out of currency, as discussed in Jiang and Shao (2014a,b).  

For price-change data we mainly use Klenow and Kryvtsov (2008), and benchmark their average duration of 8.6, but alternatives are also considered since there are differences across and within studies depending on details. Their average absolute price change is 11.3%, well above average inflation due to negative changes. Since the Klenow-Kryvtsov data are monthly, the model period is a month, and model-generated money demand is aggregated to quarterly to line up with Lucas-Nicolini. A month also seems natural since it corresponds to credit card billing period. However, this does not matter much: as usual, a convenient feature of

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14These numbers are shares of credit transactions by volume. In Canadian data the fraction by value is double, 40%, since as theory predicts credit is used for larger purchase. However, in Boston Fed data, the fractions by value and volume are about the same. There seems to be no consensus why American and Canadian data differ on value, but in any case, we use volume.
search models is that they can be fit to different frequencies simply by scaling parameters like arrival and discount rates.

### 5.2 Basic Findings

Generally, while we cannot hit all the targets exactly, we get close, except where indicated. The results are in Table 1. Consider first the fixed-cost model, which hits all targets except the fraction of credit transactions, because our parameter search is constrained to stay within the region where MME exists. Trying to get 20% BJ credit transactions forces $\delta$ into a region where MME does not exist for some of values of $i$ in the sample. Hence, for this model we use the smallest $\delta$ consistent with MME at the maximum observed $i = 0.085$, which yields 8% credit transactions. This $\delta$ is about 1.5% of the BJ utility parameter $\mu$, which comes primarily from matching average real balances. The value of $\gamma$, about half of $\mu$, comes primarily from the markup. The probability of sampling one price (two prices) in BJ is $\alpha_1 = 0.18$ ($\alpha_2 = 0.23$).

<table>
<thead>
<tr>
<th></th>
<th>BJ utility $\mu$</th>
<th>BJ cost $\gamma$</th>
<th>credit cost $\delta/\tau$</th>
<th>$pr(n = 1)$ $\alpha_1$</th>
<th>$pr(n = 2)$ $\alpha_2$</th>
<th>tie-break $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fix</td>
<td>104.06</td>
<td>57.37</td>
<td>1.643</td>
<td>0.18</td>
<td>0.23</td>
<td>0.90</td>
</tr>
<tr>
<td>Var</td>
<td>21.47</td>
<td>11.53</td>
<td>0.063</td>
<td>0.12</td>
<td>0.17</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Table 1: Baseline Calibration

For the variable-cost model, in contrast, we approximates all targets very well, including 20% for BJ credit. Note $\mu$ and $\gamma$ are lower than the fixed-cost case, so BJ goods are now less important relative to AD goods, but $\gamma/\mu$ is similar. With $E_{Hq}$ around 15.4, the average transaction cost $\tau E_{Hq}$ is about 0.97. Scaled by BJ utility, $\tau E_{Hq}/\mu = 0.045$. To judge whether this is reasonable, note the average US sales tax is 0.064. Adding average credit cards fees of around 1.5-2% (without counting the small fixed costs per transaction), this is very much in the ballpark. Also notice $\alpha_1$ and $\alpha_2$ are lower than in the fixed-cost model. However,
one constant across specifications is the tie-breaking parameter $\sigma = 0.9$, implying that indifferent sellers change prices only 10% of the time.

Figure 6 shows money demand, with the solid curve from the fixed-cost model and the dashed curve from the variable-cost model. The fit is good in both cases, although the curves are somewhat different at low values of $i$. While this difference can be important for other issues, it may not be too important for our applications. In general, we conclude that the variable-cost specification demonstrates the ability to match both money demand and micro payment data very well, while the fixed-cost model can match money demand but has trouble with the micro payment observations.

![Figure 6: Money Demand for Different Specifications](image)

6 Sticky Prices?

Section 4.3 shows the model can generate sticky prices qualitatively, but how well does it do quantitatively? Figure 7 shows the Klenow-Kryvtsov data, plus the model predictions distributions for the two specifications, where we recall that the variable-cost model hits the share of money and credit in the micro data, but the fixed-cost model does not. Still, both capture the overall shape of the empirical histogram, although the fit is not perfect. We now argue, however, that the theory
is broadly consistent with several facts deemed important in the literature.

The average absolute change is 11.5% in the fixed-cost model, 12.3% in the variable-cost model and 11.3% in the data. This is no surprise, as this is a calibration target. Statistics we did not target include the fraction of small changes (below 5% in absolute value), which is 44% in the data, 31% in the fixed- and 30% in the variable-cost model. So, on this the theory is slightly off, but not dramatically so.\textsuperscript{15} Similarly, the fraction of big changes (above 20% in absolute value) is 16% in the data, 20% in the fixed- and 23% in the variable-cost model, while the fraction of negative changes is 37% in the data and 43% in both models. So, on these theory is not too far off. It is not trivial to match these facts in other models. Our benchmark setup does fairly well, basically, because a shifting $F_t$ and a tie-breaking rule calibrated to duration generate large average, many small, many big and many negative adjustments.

Figure 8 shows the hazard (probability of changing $p$ as a function of time since the last change). The left panel is the from the data in Nakamura and Steinsson (2008), who argue this is interesting, and from the model with a variable cost

\textsuperscript{15}Eichenbaum et al. (2015) find a fraction of small prices changes lower than other studies, and suggest this is because others did not correctly correct for measurement error. Here we take the Klenow-Kryvtsov numbers at face value.
of credit. We do not generate enough action at low durations, evidently, but at least the hazard slopes downward, something Nakamura and Steinsson say is hard to get. Of course, one should not expect to explain every nuance, as there may be a lot going on that is not in the model which could increase adjustments at low durations – e.g., experimentation by sellers trying to learn market conditions. Even without this, the hazard decreases initially before turning up at around 3 years, as shown in the right panel. It is U-shaped over a longer horizon, naturally, as inflation means any $p$ leaves the moving support $\mathcal{F}_t$ eventually.\footnote{Yet even at 10 years, our hazard is only up to 14%. Hence, some firms can stick to prices for a very long time, as long as Cecchetti’s (1986) magazines mentioned in Section 2.} While there are other ways to think about these data (e.g., perhaps learning along the lines of Bachmann and Moscarini 2014), it seems worth pointing out what a simple search model predicts.

Figures 9 and 10 show the impact of changing duration and inflation in the variable-cost model. The left panel of Figure 9 is for $\sigma \simeq 0$ and an expected duration of 1 month; the chooses $\sigma$ to fit the histogram, implying $\sigma = 0.95$ and an expected duration of 16 months. Clearly, the right panel fits better than the benchmark duration of 8.6 months. With too much stickiness, the fit gets very bad: at $\sigma = 0.9999$, e.g., the fraction of negative changes drops to 1.5%. The left panel
of Figure 10 sets $\pi$ to 0, and the right to 22.5%. One can check, not surprisingly, that the fraction of negative adjustments decreases while both the frequency and size increase with $\pi$. This is consistent with the evidence, as discussed by Klenow and Kryvtsov’s (2008).

Overall, while the theory misses some details, we think it performs reasonably well. It would be hard to say there is anything puzzling about price changes in the data – it is close to what one should expect from rudimentary search theory. It would be even harder to argue there is anything definitively informative about Mankiw costs or Calvo arrivals in the data, given the outcomes we generate without such devices. We also emphasize the discipline imposed by macro and micro
observations on money and credit. If we ignore these observations, we can do better. Figure 11 shows a calibration that does not try to match money demand. The histogram fits nearly perfectly, although we are way off on money demand. A conclusion is that it is easy to capture sluggish price adjustment, but we more ambitiously can capture this reasonably well, while also accounting for the evidence on credit and money demand.

7 Welfare

The next application measures the welfare cost of inflation in MME. As is standard, we compute the percent change in consumption that is equivalent to changing $\pi$ from some level to 0. Given any $\pi$, welfare is

\[
(1 - \beta) W(\pi) = U(x^*) - x^* + (\alpha_1 + \alpha_2)(\mu - \gamma) - (\alpha_1 + \alpha_2)\left\{ \delta [1 - H_\pi (z_\pi)] + \tau \int_{z_\pi}^{\bar{q}} (q - z_\pi) dH_\pi \right\}, \quad (15)
\]

consisting of the AD surplus $U(x^*) - x^*$, the expected BJ surplus $(\alpha_1 + \alpha_2)(\mu - \gamma)$, and the expected resource cost of credit. The welfare cost of having (annual) inflation $\pi$ is the percentage of consumption households would give up to change $\pi$ to 0.
Figure 12 shows that this welfare cost monotonically increases with inflation in both models. The variable-cost model supports MME for a wider range of \( \pi \), and the right panel of Figure 12 only shows the welfare cost between the Friedman rule and 20\%. The welfare costs are small in both models. The fixed-cost model predicts that the welfare cost of 5\% annual inflation is only 0.14\% of real consumption. The variable-cost version predicts 0.1\% for 5\% and 0.2\% for 10\% annual inflation, smaller than the estimate in Lucas (2000), and much smaller than Lagos and Wright (2005). This is because in the present setup changes in inflation affects neither the intensive margin of BJ trade – the good is indivisible – nor the extensive margin – the population of participants is fixed. Hence, the welfare cost comes from the increased cost of using credit.

In the fixed-cost model, inflation has no effect on the real price distribution, but buyers substitute out of money and into credit as \( \pi \) rises, and greater use of BJ credit entails a larger resource cost. In the variable-cost model, inflation affects \( G(q) \) by changing \( \bar{q} \), implying two effects on welfare. On the one hand, when \( \pi \) increases, buyers not only use credit more often, they also use more credit to pay for BJ goods. On the other hand, households hold less cash as it becomes more costly, which lowers the maximum amount they can pay for BJ goods. In equilibrium, real BJ prices decrease with \( \pi \), and so does the total variable cost of credit. On net, the welfare cost increases with inflation, but at a slower rate than the fixed-cost model.

We also ask about the relation between inflation, markups and price dispersion. In the fixed-cost model, since the distribution of real BJ prices does not depend on \( \pi \), both the average markup and dispersion in the BJ market do not change with \( \pi \). In the variable-cost model, \( G(q) \) decreases with \( \pi \) in the sense of first-order stochastic dominance. Consequently, both the average markup and dispersion, measured by the coefficient of variation, decrease with \( \pi \), as shown in Figure 13. Both \( \bar{q} \) and \( q \) decrease with \( \pi \), and \( \bar{q} \) falls faster. As we said earlier, Ben-
abou (1992) finds a small but significant negative relation between markups and inflation. Parsley (1996) and Debelle and Lamont (1997) find a positive relation between inflation and price dispersion. Reinsdorf (1994) argues that inflation and price dispersion may be negatively related, and Caglayana et al. (2008) presents a U-shaped relationship. So, while the facts may not be unequivocally established, the theory does speak to the issues.

8 Endogenous Participation

The next application endogenizes participation by buyers in the BJ market, which is interesting relative to the benchmark model, because it allows us to make other
points about sticky prices. Let \( W^1(A) \) and \( W^0(A) \) be the AD value functions for households that enter and skip the next BJ market, resp. Then \( W(A) = \max \{W^1(A), W^0(A)\} \), and \( W(A) = W^1(A) \) as long as some buyers enter. If \( k \) is the entry cost, then (7) changes only slightly to

\[
W(A) = A + U(x^*) - x^* - k + \beta \max_z O_i(z). \tag{16}
\]

For some but not all households to enter BJ, \( b \in (0, \bar{b}) \), we need \( \beta \Phi = k \), where recall the surplus from participation is

\[
\Phi \equiv (\alpha_1 + \alpha_2) [\mu - \mathbb{E}_H q - \tau \mathbb{E}_H \max (0, q - z)] - i \hat{\gamma}_i. \tag{17}
\]

Assume no that the BJ arrival rates depend on the buyer-seller ratio (or market tightness), \( \alpha_n = \alpha_n(b_t) \). With \( k = 0 \), in the benchmark model, \( \alpha_n(\bar{b}) \) is fixed. Since \( y \) is indivisible, BJ output is fixed, too, and AD output is fixed by \( U'(x) = 1 \). Hence changes in the level of \( M \), or in \( \pi \) and \( i \), have no impact on output (although, as discussed above, changes in \( i \) or \( \pi \) affect the mix between money and credit and involves resource costs). With \( k > 0 \), however, \( b \) adjusts until the marginal entrant is indifferent, endogenizing BJ output along the extensive margin. Now changes in \( i \) or \( \pi \) affect output, but of course a one-time unanticipated change in \( M \) does not. Changes in \( M \) simply change \( \phi \) so that real variables, like \( \phi M \) and the distribution \( G(q) \), stay the same – this is classical neutrality.

For the sake of illustration, suppose that each period buyers in BJ attempt to solicit two price quotes, and succeed in each try independently with probability \( s = s(b) \), depending on market tightness, where \( s(0) = 1 \), \( s(\bar{b}) = 0 \), \( s'(b) < 0 \), and \( s''(b) > 0 \), as is many search models. Then \( \alpha_1(b) = 2s(b)[1 - s(b)] \) and \( \alpha_2(b) = s(b)^2 \).

Using these, we get a special case of the money demand functions for the fixed- and variable-cost models, defining relation between \( \hat{z}_i \) and \( b \) called the RB (real balance) curve. Then \( \beta \Phi = k \) defines a relation called the FE (free entry) curve. The intersection of the RB and FE curves constitutes a steady state MME at \((\hat{z}_i^*, \bar{b}^*)\).
As shown in Figure 14, RB is increasing and convex, with \( z_i = \gamma \) at \( b = 0 \), while FE slopes up (down) to the left (right) of RB, with \( b \in (0, \bar{b}) \) at \( z_i = 0 \). Therefore there is a unique MME at \((\hat{z}_i^*, \hat{b})\), and given this, \( F(p) \) and \( G(q) \) are constructed as usual. One can now work out the effects of changes in exogenous variables. It is already obvious that (surprise, one-time) changes in \( M \) are neutral. An increase in \( i \), however, shifts both curves toward the origin and output falls. Monetary policy, in the sense of the nominal interest rate \( i \), or the inflation rate \( \pi \), matters; this is not because prices are sluggish, but because higher \( i \) or \( \pi \) means a higher tax on decentralized exchange.\(^{17}\)

![Figure 14: Real Balances and Free Entry](image)

In contrast, if prices were sticky for Calvo or Mankiw reasons, a one-time unanticipated jump in \( M \) generally has real effects. This is because at least some firms do not adjust \( p \), even though they would like to, and hence the nominal distribution \( F(p) \) does not change enough to keep the same real distribution \( G(q) \). Hence, the real distribution can turn in favor of buyers, increasing \( b \), and thus stimulating output, under the standard assumption that firms sell to anyone that

\(^{17}\)We will not dwell on existence or uniqueness in this extended model, because we use it mainly to make one simple point. However, to give some intuition, there should be no presumption of multiplicity here, as there would be if instead we had entry by sellers. With seller entry, as their number increase, buyers increase real balances, and vice versa. Models that are related, except they use bargaining or competitive search, include Rocheteau and Wright (2005,2015), with seller entry, and Liu et al. (2013), with buyer entry.
shows up. By contrast, in our economy, a jump in $M$ affects neither $G(q)$ nor $b$. A policy advisor seeing only a fraction of sellers adjusting $p$ each period may conclude that jumps in $M$ would have real effects; that would be wrong. But of course, we do have affects from other policy variables, including $i$ and $\pi$. The bigger point may be that the underlying reason why prices are sticky can make a difference for policy analysis.

9 Dynamics

As a final application, we consider nonstationary equilibrium. To make a point, we also go beyond the case of fiat currency by allowing $m$ to potentially earn a flow return $\rho$. If $\rho > 0$ then, as in standard finance, asset $m$ can be interpreted as a share in a technology (‘tree’) bearing a dividend (‘fruit’) in each AD market in units of numeraire. If $\rho < 0$ it can be interpreted as an asset-holding or storage cost, as in models of commodity money. Here we keep the asset supply $M$ fixed over time, $\pi = 0$. In fact, it is easy to relax this, as we did above for fiat money, but one should probably not think of $\pi$, the growth rate of $M$, as a policy choice when $\rho \neq 0$. We also revert to $k = 0$, so $b = \bar{b}$ and the $\alpha$’s are fixed.

The household problem is

$$W(A) = A + U(x^*) - x^* + \beta \max_z O_i(z)$$

where $A = \rho m + \phi m - \phi d - C(d) + T$ now includes $\rho m$, and $O_r(z) = V(z) - (1 + r)z$ has $i$ replaced by $r$ since $\pi = 0$. The FOC $V'(z) = 1 + r$ easily yields the Euler equation, which is for the preferred variable-cost model

$$\phi_t = \beta (\phi_{t+1} + \rho) \left[ 1 + (\alpha_1 + \alpha_2) \tau \hat{H}(\hat{z}) \right],$$

where we do not impose stationarity of the asset price $\phi_t$. If there were no BJ market (say, $\alpha_1 = \alpha_2 = 0$) this would look like a standard asset-pricing equation given our AD utility function, $\phi_t = \beta (\phi_{t+1} + \rho)$; in general it is augmented on the
RHS by the liquidity value of the asset, which is the expected reduction in costly credit.

Inserting the equilibrium \( H(\tilde{z}) \), after routine algebra, we get

\[
\phi_t = \beta \left( \phi_{t+1} + \rho \right) \left[ 1 + \frac{\tau \alpha_1^2 \left[ \mu - \left( \rho + \phi_{t+1} \right) \right]}{4\alpha_2} \frac{\left[ \mu + \left( \rho + \phi_{t+1} \right) (1 + 2\tau) - 2\theta (1 + \tau) \right]}{(1 + \tau)^2 \left( \rho - \gamma + \phi_{t+1} \right)} \right]
\]

This dynamic system gives the asset price today in terms of its price tomorrow, say \( \phi_t = \Phi(\phi_{t+1}) \). The left panel of Figure 15 shows \( \phi_t = \Phi(\phi_{t+1}) \), as well as the inverse \( \phi_{t+1} = \Phi^{-1}(\phi_t) \), for the calibrated parameters, including \( \rho = 0 \), except now \( \pi = 0 \) and hence \( i = r \). There is a unique steady state MME at around \( \phi = 13 \); it looks like there is another steady state around \( \phi = 10 \) but that is not an equilibrium because the calibrated cost is \( \gamma = 11.47 \), and hence sellers make negative profit. Here the steady state is unstable.

The right panel of Figure 15 makes only one change in parameter values: \( \alpha_1 \) is reduced from 0.12 to 0.02. There is still a unique steady state MME with \( \phi \) around 12, because again the lower intersection of \( \Phi \) with the 45° line has \( \phi < \gamma \) and hence profit is negative. Using standard methods of dynamic theory (e.g., Azariadis 1993), because we now have \( \Phi' < -1 \) at the steady state, \( \Phi \) and \( \Phi^{-1} \) cross off the 45° line, and hence there exists a cycle of period 2. Over the cycle, the asset price \( \phi \) oscillates between \( \phi_L \) and \( \phi_H \) as a self-fulfilling prophecy. It is not
atypical for monetary models to display this kind of dynamic equilibrium, except for one feature: as φ fluctuates, so does the entire price distribution F(p), as given by (3). Also, these are not purely nominal fluctuations – they matter for payoffs.

For the same parameters that generate the 2-cycle, Figure 16 shows the third iterate \( \Phi^3(\phi) \). It has 6 intersections with the 45° line in addition to the steady state. This means there exist a pair of 3-cycles. Standard results tell us that the existence of a 3-cycle implies the existence of \( N \)-cycles for all \( N \) by the Sarkovskii theorem, as well as chaotic dynamics by the Li-Yorke theorem (see Azariadis 1993). We conclude that the model can in principle generate a large set of perfect foresight dynamics, and although that is not the case at the calibrated parameter values, it is the case for parameters close to calibrated values. There are also sunspot equilibria for similar parameter values.\(^{18}\)

For \( \rho = 0 \), one might think it is natural to get multiplicity, since \( \phi > 0 \) and \( \phi = 0 \) are both steady state equilibria. But that would be incorrect since we can eliminate the nonmonetary equilibrium by setting \( \rho > 0 \), and as long as \( \rho \) is not too big the qualitative results are similar. One should not think of these cycles or sunspots as fluctuating across two steady states but as fluctuating around a steady state. At the same time, by setting \( \rho < 0 \) we can get multiple steady states, as shown in the right panel of Figure 16, which uses the same parameters as above except it has \( \rho = -0.4 \). In this situation, the lower steady state is stable. Given a stable steady state there are other types of sunspot equilibria.\(^{19}\)

Again, these results concern dynamics of the entire price distribution \( F(p) \).

---

\(^{18}\)A sunspot equilibrium is one where the outcome fluctuates randomly even when fundamentals are constant. A standard method of proving sunspot equilibria exist, going back to Azariadis and Guesnerie (1986), is to suppose the outcome depends on an extrinsic two-state Markov process, \( s \in \{s_1, s_2\} \), where \( \varepsilon_s = \text{prob}(s_{t+1} \neq s|s_t = s) \). If \( \varepsilon_1 = \varepsilon_2 = 1 \) this reduces to a 2-cycle, the existence of which we proved by example. By continuity there are sunspot equilibria at least for \( \varepsilon_s \) close to 1.

\(^{19}\)We seek \((\phi_1, \phi_2, \varepsilon_1, \varepsilon_2)\) such that \( \phi_1 = \varepsilon_1 \Phi(\phi_2) + (1 - \varepsilon_1) \Phi(\phi_1) \) and \( \phi_2 = \varepsilon_2 \Phi(\phi_1) + (1 - \varepsilon_2) \Phi(\phi_2) \), where \( \varepsilon_s \in (0, 1) \) and wlog \( \phi_2 > \phi_1 \). These equations are easy to solve for \( \varepsilon_1 \) and \( \varepsilon_2 \). Whenever \( \Phi'(\phi_s) > 1 \) at a steady state \( \phi_s \), for any \( \phi_1 \) in some range to the left of \( \phi_s \) and any \( \phi_2 \) in some range to the right of \( \phi_s \) it is easy to check \( \varepsilon_1, \varepsilon_2 \in (0, 1) \).
Figure 16: Examples with a 3-Cycle and with Two Steady States

Our sellers are perfectly rational and forward looking in these dynamic equilibria. This is not always the case in similar models that use bargaining, where axiomatic solution concepts are used. Axiomatic bargaining solutions sometimes can be rationalized as corresponding to strategic models, as is the well-known case of Nash emerging as the limit of an alternating-offer game, at least in stationary situations. But it is not trivial to generalize those kinds of limiting arguments to nonstationary equilibria. When one applies the methods in Binmore et al. (1986), e.g., to nonstationary settings with nonlinear utility, it can be shown that the outcome corresponds to Nash bargaining in but not out of steady state. Moreover, it can be argued that using Nash out of steady state is tantamount to using the extensive-form game with myopic agents, who negotiate as if conditions were constant, even while they are not (see Burdett et al. 2015 for citations and further discussion). This is another reason to prefer price posting over bargaining.

The present model generates interesting dynamics, where an entire distribution $F(p)$ varies as a self-fulfilling prophecy with agents that are fully rational and forward looking, and yet it is not particularly difficult to handle analytically. In cyclic, chaotic or stochastic (sunspot) equilibrium, the use of money and credit vary over time. There are in the literature several attempts to analyze credit cycles, includ-
ing Kiyotaki and Moore (1997), Gu et al. (2013) and references therein. Here cycles where the use of credit varies over time emerge endogenously. Moreover, apropos our discussion of sticky prices, notice the following: Over time as $F(p)$ varies, as long as there is overlap in the supports, there are equilibria where some sellers stick to their prices in the face of changing economic conditions. The intuition is similar – e.g., when $F(p)$ shifts, some sellers stick to their nominal $p$ because there is simply no $\hat{p}$ that they can pick that constitutes a profitable deviation. The point is simply that the logic applies not only to steady and perfectly anticipated inflation; it applies given any of a variety of changes in economic conditions.

## 10 Conclusion

This paper has explored models of money and credit as alternative payment methods. A key component was that we used price posting, as in Burdett-Judd, to determining the terms of trade. This was integrated into the monetary model of Lagos-Wright, extended to allow costly credit, which helped resolve an indeterminacy problem in other models with money and posting. The resulting framework has firms that charge different prices, buyers that use cash (credit) for less (more) expensive purchases, and nominal stickiness. It is still quite tractable, and for both fixed- and variable-cost specifications, we provide clean results about the existence, and uniqueness or multiplicity, of different types of stationary equilibria, including those where agents use money or credit or both. We also describe nonstationary equilibria, including cyclic, chaotic and stochastic (sunspot) equilibria. In fact, both the fixed- and variable-cost specifications delivered closed-form solutions for money demand that resemble Baumol-Tobin, in terms of algebra and in terms of economics, and fit the macro data well. However, only the variable-cost model was able to match the share of credit in the micro data.

The models can fit the price change data extremely well when we did not impose the discipline of matching the money demand observations; and even when
we did impose this discipline, the model fit the price change data fairly well. We do not mean it fit in a statistical sense; we merely mean that the theory generates outcomes that are roughly consistent with the salient facts, some of which are deemed puzzling in the literature (recall fn. 4). It is also important to emphasize the nature of our claim regarding price-change behavior. We are not saying that the model has a unique prediction that lines up with the data; it does not. We are simply saying that there is an equilibrium on the model that can account for this behavior. In other words, to repeat what we said above, the exercise shows by example that there may be less information in the data – e.g., concerning Mankiw-style menu costs or Calvo-style arrival rates – than one might have thought. We also believe it is relevant to consider a model that is jointly consistent with the money demand, credit share and price change observations because they are three standard observations concerning nominal magnitudes. Why not try to account for them all in one setting?

In summary, there is much in this paper that is not new. The basic Lagos-Wright structure has been around for over a decade, Burdett-Judd pricing much longer, and there have been a few papers that combine the two, as discussed in Section 2. There is a huge literature on money and credit that appropriately comes in many interesting varieties, including Lucas-Stokey models to analyses using the mechanism design approach surveyed in Lagos et al. (2015). Still we think our combination of ingredients is worth consideration for no other reason than we have shown it to be useful in several different applications. It is certainly tractable, it generates arguably realistic outcomes like the coexistence of money and credit, and prices that are definitely disperse and potentially sticky, it is amenable to easy calibration, at least the variable-cost specification can match what we consider the salient observations, and it allows some interesting new perspectives on policy. In particular, we used it to organize observations on how markups, dispersion, and welfare are related to inflation. We also used it to show how interesting endogenous
dynamics emerge in a rigorous yet tractable way. We believe other insights will emerge from additional applications in future work.
Appendix

Derivation of (8): Consider $\delta > 0 = \tau$ to reduce notation. Then write

$$V(z) = W(z + T) + \alpha_1(b) \int_{\tilde{q}}^{q} (\mu - q) dG_1(q) + \alpha_1(b) \int_{\tilde{q}}^{q} (\mu - q - \delta) dG_1(q)$$

$$+ \alpha_2(b) \int_{\tilde{q}}^{q} (\mu - q) dG_2(q) + \alpha_2(b) \int_{\tilde{q}}^{q} (\mu - q - \delta) dG_2(q),$$

where $G_n(q) = 1 - \hat{G}(q)^n$ is the CDF of the lowest of $n$ draws from $G(q)$. The first term is the continuation value if a buyer does not trade. The second is the probability of meeting a seller with $q \leq z$, so only cash is used, times the expected surplus, which is simple because $W'(A) = 1$. The third is the probability of meeting a seller with $q > z$, so credit must be used, which adds cost $\delta$. The last two terms are similar except the buyer meets two sellers. The rest is algebra.

Proof of Lemma 3: (i) In NME, buyers’ BJ surplus is $\Sigma = \mu - q - \delta - \tau q$. Note $\Sigma = 0$ at $q = (\mu - \delta) / (1 + \tau)$, so no buyer pays more than this. If $\tilde{q} < (\mu - \delta) / (1 + \tau)$ then the highest price seller has profitable deviation toward $(\mu - \delta) / (1 + \tau)$, which increases profit per unit without affecting sales. Hence $\tilde{q} = (\mu - \delta) / (1 + \tau)$. (ii) In MME, for $q > z$, as it is near $\tilde{q}$, $\Sigma = \mu - q - \delta - \tau(q - z)$. Note $\Sigma = 0$ at $q = (\mu - \delta + \tau z) / (1 + \tau)$, and repeat the argument for NME to show $\tilde{q} = (\mu - \delta + \tau z) / (1 + \tau)$. The definition of MME has $z < \tilde{q} = (\mu - \delta + \tau z) / (1 + \tau)$, which reduces to $z < \mu - \delta$. (iii) In PME, given buyers bring $z$ to BJ, they would pay $z$. Hence $\tilde{q} \geq z$, as $\tilde{q} < z$ implies the highest price seller has profitable deviation. We also have to be sure there is no profitable deviation to $q > z$, which requires buyers using some credit. The highest such $q$ a buyer would pay solves $\Sigma = \mu - q - \delta - \tau(q - z) = 0$, or $q = (\mu - \delta + \tau z) / (1 + \tau)$. There is no profitable deviation iff $(\mu - \delta + \tau z) / (1 + \tau) \leq z$, which reduces to $z \geq \mu - \delta$. ■

Proof of Proposition 1: (i) With fiat currency $\phi = 0$ is always self-fulfilling, so we simply set $G(q)$ according to (5), corresponding to equilibrium in the original BJ model.

(ii) From Figure 1, MME exists iff three conditions hold: (a) $O_i^-(\tilde{q}) < 0$; (b) $O_i^+(\tilde{q}) > 0$; and (c) $O_i(z_i) > O_i(0)$. Now (a) is equivalent to $(\alpha_1 + \alpha_2) \delta H^-(\tilde{q}) < i$, which holds iff $i > \tilde{i}$. Then (b) is equivalent to $(\alpha_1 + \alpha_2) \delta H^+(\tilde{q}) > i$, which holds iff $i < \tilde{i}$ where $\tilde{i} = \delta (\alpha_1 + 2\alpha_2)^3 / 2\alpha_1\alpha_2 (\mu - \delta - \gamma) > i$. Also, (c) is equivalent to
\((\alpha_1 + \alpha_2) \delta H (z_i) - iz_i > (\alpha_1 + \alpha_2) \delta H (0)\), which holds iff \(\Delta (i) > 0\) where

\[
\Delta (i) = -i\gamma + \frac{\delta (\alpha_1 + 2\alpha_2)^2}{4\alpha_2} - i^{\frac{2}{3}}\delta^{\frac{1}{3}}\alpha_1^{\frac{2}{3}}\alpha_2^{-\frac{1}{3}} (\mu - \delta - \gamma)^{\frac{2}{3}} (2^{-\frac{1}{3}} + 2^{-\frac{2}{3}}).
\]

Notice \(\Delta (0) > 0 > \Delta (\bar{i})\) and \(\Delta (i) > 0\) iff \(i < \bar{i}\). It remains to verify that \(\bar{i} > i\), so that (a) and (c) are not mutually exclusive. It can be checked that this is true iff \(\delta < \bar{\delta}\). Hence a MME exist under the stated conditions. It is unique because \(\bar{q} = \mu - \delta\), which pins down \(G (q)\), and then \(\hat{z}_i = \arg \max_{z \in (q, \bar{q})} O_i (z)\).

(iii) From Figure 1, PME exists iff three conditions hold: (a) \(O_i^- (\bar{q}) > 0\); (b) \(O_i^+ (\bar{q}) > 0\); and (c) \(O_i (\bar{q}) > O_i (0)\). Now (a) holds iff \(i < i\) and (b) holds iff \(i < \bar{i}\). Condition (c) holds iff \(i < \bar{i}\). For \(\delta > \bar{\delta}\), it can be checked that \(\bar{i} > i\) and \(\bar{i} < \bar{i}\), so the binding condition is \(i < \bar{i}\). For \(\delta < \bar{\delta}\), it is easily checked that \(\bar{i} < \bar{i}\), and \(\bar{i} < \bar{i}\), so the binding condition is \(i < \bar{i}\).

**Formulae for Calibration:** Consider the fixed-cost model. Inserting \(\bar{q}\) and \(q\), we reduce the posted-price and transaction-price CDF’s to

\[
G (q) = 1 - \frac{\alpha_1 (\mu - \delta - q)}{2\alpha_2 (q - \gamma)},
\]

\[
H (q) = 1 - \frac{\alpha_2^2 (\mu - \delta - q) (\mu - \delta + q - 2\gamma)}{4\alpha_2 (\alpha_1 + \alpha_2) (q - \gamma)^2}.
\]

The the fraction of monetary transactions and the markup are

\[
H (\hat{z}_i) = \frac{(\alpha_1/2 + \alpha_2)^2 - [\alpha_1 \alpha_2 (\mu - \delta - \gamma)]^{2/3} i^{2/3}}{\alpha_2 (\alpha_1 + \alpha_2) (4\delta)^{2/3}}
\]

\[
\frac{\mathbb{E}_{G(q)}}{\gamma} = 1 + \frac{\alpha_1 (\mu - \delta - \gamma) \log (1 + 2\alpha_2/\alpha_1)}{2\alpha_2 \gamma}.
\]

The Lucas-style money demand function and its elasticity are

\[
L_i = \frac{\gamma + [\alpha_2^2 \delta (\mu - \delta - \gamma)^2 / 2\alpha_2]^{1/3} i^{-1/3}}{1 + \alpha_1 (\mu - \delta) + \alpha_2 \gamma} - 1
\]

\[
\eta_i = \frac{3 + 3 \gamma \left[\alpha_2^2 \delta (\mu - \delta - \gamma)^2 / 2\alpha_2\right]^{-1/3} i^{1/3}}{3 + 3 \gamma \left[\alpha_1^2 \delta (\mu - \delta - \gamma)^2 / 2\alpha_2\right]^{1/3} i^{-1/3}}.
\]

The variable cost case is similar.
References


