On the Benefits of Costly Voting*

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Abstract

We study strategic voting in a Condorcet type model in which voters have identical preferences but differential information. Voters incur private costs of going to the polls and may abstain if they wish; hence voting is voluntary. We show that in large majority elections, there exists a unique equilibrium. In this equilibrium, voting is sincere. Thus, in contrast to situations with compulsory voting, there is no conflict between strategic and sincere behavior. Furthermore, participation rates are such that in the limit, the correct candidate is elected with probability one. Finally, we show that in large elections, voluntary voting is welfare superior to compulsory voting.

1 Introduction

Condorcet’s celebrated Jury Theorem states that, when voters have common interests but differential information, sincere voting under majority rule produces the correct outcome in large elections. There are three key components to the theorem. First, it postulates that voting is sincere—that is, voters vote solely according to their private information. Recent theoretical work shows, however, that sincerity is inconsistent with rationality—it is typically not an equilibrium to vote sincerely. The reason is that rational voters will make inferences about others’ information and as a result sometimes will have the incentive to vote against their own private information (Austen-Smith and Banks, 1996).

The second component is the voting rule itself. Under a supermajority voting rule, even sincere voting will produce the wrong outcome if voters’ signals, while informative, are not of sufficiently high quality. Interestingly, the failure of the first component negates the failure of the second—once rational behavior is postulated, a generalized form of the Jury Theorem holds for all voting rules (save for unanimity). In large elections, there exist equilibria in which the correct candidate is always chosen

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(Feddersen and Pesendorfer, 1997 & 1998). Equilibrium behavior, however, involves the use of mixed strategies which are sensitive to both the population size and the particular voting rule. As a consequence, these convergence results, while powerful, may be criticized as relying on voting behavior that seems implausible.

Moreover, these generalizations of the Jury Theorem continue to rely on its third component—the assumption that voter turnout is high. Indeed it is implicitly assumed that voting is compulsory, so low turnout is not an issue. When voting is voluntary and costly, however, there is reason to doubt that voters will turn out in large enough numbers to guarantee correct choices. Indeed rational voters will correctly realize that a single vote is unlikely to affect the outcome. This is the so-called “paradox of not voting” (Downs 1957).

In this paper, we revisit the classic Condorcet Jury Theorem but with two amendments to the environment. First, we relax the assumption that the size of the electorate is fixed and commonly known in favor of one where the size is a Poisson random variable as in the model introduced by Myerson (1998 & 2000). This, by itself, affects very little of the findings discussed above but as Myerson (1998) has demonstrated, leads to a simpler analysis. Second, and more important, we relax the (implicit) assumption that voting is compulsory. Specifically, voters incur private costs of voting and may avoid these by abstaining. Voters in our model are fully rational, so the twin problems of strategic voting and the paradox of not voting are present.

Our main result is that under majority rule, in any large election, there is a unique equilibrium. This equilibrium has the following features:

1. Voting is sincere. Thus when voting is costly and voluntary, there is no conflict between rationality and sincerity.

2. Regardless of the quality of voters’ information, the equilibrium participation rates are such that the correct candidate is chosen.

3. Even though the turnout percentage goes to zero in the limit, the number of voters is unbounded—regardless of the distribution of voting costs.

To summarize, we show that if the classic Condorcet model is amended to allow for costly voting and voluntary participation, then many desirable features of the original Jury Theorem may be restored—sincere voting obtains as an equilibrium phenomenon and the correct candidate is always chosen in the limit.

To see why voluntary (and costly) voting leads to sincere behavior, consider the following situation. Suppose that voters have 50-50 prior beliefs as to which is the correct candidate and the election will be decided according to majority rule. Further suppose that signals in favor of \( A \) are more accurate than those in favor of \( B \). In other words, a signal in favor of \( A \) is more likely in situations in which \( A \) is the correct candidate (say the chances of this are 75%) than a signal in favor of \( B \) is in situations in which \( B \) is the correct candidate (say the chances of this are 60%).

First, suppose there is a fixed and known (large) population and voting is compulsory. Suppose further that all voters save one, vote sincerely and consider a voter with a signal in favor of \( A \). This voter is pivotal—his vote affects the outcome—if
the vote counts are roughly equal. But since signals for A are more accurate than signals for B, a roughly 50-50 vote split is more likely when B is the right candidate. Thus a voter with signal A should rationally vote for B. It is not an equilibrium for everyone to vote sincerely.

Now suppose that voting is voluntary and participation behavior is such that those with information favorable to B are more likely to turn out than those with information favorable to A. Now the fact that the votes are roughly the same does not imply that the signals are biased in favor of B. Now those voters in favor of B are more likely to vote and this fact mitigates the biased inference from the split vote itself. Our main result exploits this reasoning and shows that in fact the endogenously determined participation rates are such that the resulting inferences lead to sincere voting behavior.

**Related literature** Early work on the Condorcet Jury Theorem viewed it as a purely statistical phenomenon—perhaps the way that Condorcet himself viewed it. When voters’ information is independently distributed it is just an expression of the law of large numbers. Ladha (1992) examines extensions of the result to the case of correlated information.

Game theoretic analyses of the Jury Theorem originate in the work of Austen-Smith and Banks (1996). They provide necessary and sufficient conditions for sincere voting to be an equilibrium. These conditions are “non-generic.” Specifically, only for very special combinations of the parameters—the prior distribution of values, the precision of voters’ information and the voting rule itself—is it the case that sincere voting constitutes an equilibrium. One such non-generic case is where voters’ belief as to which is the correct candidate is 50-50, the precision of voters’ information is symmetric with respect to the candidates and the election is decided by majority rule (with ties resolved by a fair coin). A departure from this specification along any one dimension destroys the incentives to vote sincerely. The key idea is that rational voters would make choices based on the assumption that they are pivotal—in all other cases the choice is inconsequential—and so would infer others’ information from this fact. In generic cases, equilibrium cannot involve sincere voting.

Feddersen and Pesendorfer (1998) derive the (“insincere”) equilibria of the voting games specified above—these involve mixed strategies—and then study their limiting properties. They show that, despite the fact that sincere voting is not an equilibrium, large elections still aggregate information correctly. Quite remarkably, this result holds for all supermajority rules, the only exception being the unanimity rule used for jury decisions. Feddersen and Pesendorfer (1996) contains another version of the same idea but for majority rule. McLennan (1998) views such voting games, in the abstract, as games of common interest and argues on that basis that there are always Pareto efficient equilibria of such games. Apart from the fact that voting is costly and voluntary, our basic setting is the same as that in these papers—there are two candidates, voters have common interests about differential information.

All of this work postulates a fixed and commonly known population of voters. Myerson (1998 & 2000) argues that precise knowledge of the number of eligible voters
is an idealization at best, and suggests an alternative model in which the size of the electorate is a Poisson random variable. He shows that this specification leads to a simpler analysis and derives the mixed equilibrium for the majority rule (in a setting where signal precisions are asymmetric). He then studies its limiting properties as the number of expected voters increases, exhibiting information aggregation results parallel to those derived in the known population models. As mentioned above, in this paper we also find it convenient to adopt Myerson’s Poisson game technology.

A separate strand of the literature is concerned with costly voting and endogenous participation but in settings in which voter preferences are diverse (sometimes referred to as “private values”). Palfrey and Rosenthal (1985) consider costly voting with privately known costs but where preferences over outcomes are commonly known (see also Palfrey and Rosenthal, 1983 and Ledyard, 1984 for models in which the costs are also common knowledge). These papers are interested in formalizing Downs’ paradox of not voting. Börgers (2004) studies majority rule in a costly voting model with private values—that is, with diverse rather than common preferences. He compares voluntary and compulsory voting and argues that individual decisions to vote or not do not properly take into account a “pivot externality”—the casting of a single vote decreases the value of voting for others. As a result, participation rates are too high relative to the optimum and a law that makes voting compulsory would only worsen matters. Krasa and Polborn (2007) show that the externality identified by Börgers’ is sensitive to his assumption that the prior distribution of voter preferences is 50-50. With unequal priors, under some conditions, the externality goes in the opposite direction and there are social benefits to encouraging increased turnout via fines for not voting.

Ghosal and Lockwood (2007) reexamine Börgers’ result when voters have more general preferences—including common values—and show that it is sensitive to the private values assumption.

This paper is organized as follows. In Section 2 we introduce the basic environment and review Myerson’s Poisson methodology. As a benchmark, in Section 3 we consider the model with compulsory voting and establish that sincere voting is not an equilibrium. In Section 4, we introduce the model with voluntary and costly voting. We show first that under the assumption of sincere voting, there exist positive equilibrium participation levels. We then show that given those participation levels, sincere voting is incentive compatible in large elections. Section 5 studies the limiting properties of the equilibria considered in the previous section—it is shown that in the limit, information fully aggregates and the correct candidate is elected with probability one. Finally, in Section 6 we show that there is only one equilibrium—this result uses the information aggregation property of all participation equilibria.

2 The Model

There are two candidates (or proposals), named A and B, who are competing in an election (or referendum).
There are two equally likely states of nature, $\alpha$ and $\beta$. Candidate $A$ is the better choice in state $\alpha$ while candidate $B$ is the better choice in state $\beta$. Specifically, in state $\alpha$ the payoff of any citizen is 1 if $A$ is elected and 0 if $B$ is elected. In state $\beta$, the roles of $A$ and $B$ are reversed.

The size of the electorate is a random variable which is distributed according to a Poisson distribution with mean $n$. Thus the probability that there are exactly $m$ eligible voters (or citizens) is

$$g(m, n) = \frac{e^{-n}n^m}{m!}$$

Prior to voting, every citizen receives a private signal $S_i$ regarding the true state of nature. The signal can take on one of two values, $a$ or $b$. The probability of receiving a particular signal depends on the true state of nature. Specifically,

$$\Pr[a \mid \alpha] = r \quad \text{and} \quad \Pr[b \mid \beta] = s$$

We suppose that both $r$ and $s$ are greater than $\frac{1}{2}$, so that the signals are informative and less than 1, so that they are noisy. Thus, signal $a$ is associated with state $\alpha$ while the signal $b$ is associated with $\beta$.

Conditional on the state of nature, the signals of the voters are realized independently. The posterior probabilities of the states after receiving signals are

$$q(\alpha \mid a) = \frac{r}{r + (1 - s)} \quad \text{and} \quad q(\beta \mid b) = \frac{s}{s + (1 - r)}$$

We assume, without loss of generality, that $r \geq s$. It may be verified that

$$q(\alpha \mid a) \leq q(\beta \mid b)$$

Thus the posterior probability of state $\alpha$ given signal $a$ is smaller than the posterior probability of state $\beta$ given signal $b$ even though the “correct” signal is more likely in state $\alpha$.

The election will be decided by a simple majority of the votes cast. In the event of a tied vote, the winning candidate is chosen by a fair coin toss.

### 2.1 Pivotal Events

An event is a pair of vote totals $(j, k)$ such that there are $j$ votes for $A$ and $k$ votes for $B$. An event is pivotal for $A$ if a single additional vote for $A$ will affect the outcome of the election and we denote the set of such events by $\text{Piv}_A$. One additional vote for $A$ makes a difference only if either (i) there is a tie and one more vote for $A$ will break the tie; or (ii) $A$ has one vote less than that needed for a tie and one more vote for $A$ will throw the election into a tie. Let $T = \{(k, k) : k \geq 0\}$ denote the set of ties and let $T_{-1} = \{(k - 1, k) : k \geq 1\}$ denote the set of events in which $A$ is one vote short of a tie.

Similarly, $\text{Piv}_B$ is defined to be the set of events which are pivotal for $B$. This set consists of the set $T$ of ties together with events in which $A$ has one vote more than that needed for a tie. Let $T_{+1} = \{(k, k - 1) : k \geq 1\}$ denote the set of events in which $A$ has one more vote than that needed for a tie.
2.2 Pivot Probabilities

Let $\sigma_A$ be the probability that a citizen will cast a vote for $A$ in state $\alpha$ and let $\sigma_B$ be the probability that a citizen will vote for $B$ in state $\alpha$. Analogously, let $\tau_A$ and $\tau_B$ be the probabilities that a voter will vote for $A$ and $B$, respectively, in state $\beta$.

Since it may be possible for voters to abstain, it is only required that $\sigma_A + \sigma_B \leq 1$ and $\tau_A + \tau_B \leq 1$.

Consider an event where (other than voter 1) the realized electorate is of size $m$ and there are $k$ votes in favor of $A$ and $l$ votes in favor of $B$. The number of abstentions is thus $m - k - l$. The probability of this event in state $\alpha$ is

$$\Pr[(k, l; m) | \alpha] = \frac{e^{-n} n^m}{m!} \binom{m}{k + l} (n (1 - \sigma_A - \sigma_B))^{m-k-l} (n\sigma_A)^k (n\sigma_B)^l$$

Notice that in the formula above, the expected number of votes for $A$ is $n\sigma_A$, the expected number of votes for $B$ is $n\sigma_B$ and the expected number of abstentions is $n(1 - \sigma_A - \sigma_B)$. It is useful to rearrange the expression as follows:

$$\Pr[(k, l; m) | \alpha] = e^{-(n - n\sigma_A - n\sigma_B)(n (1 - \sigma_A - \sigma_B))^{m-k-l}} \frac{(n\sigma_A)^k}{k!} e^{-n\sigma_A} \frac{(n\sigma_B)^l}{l!}$$

Of course, the size of the electorate is unknown to the voter at the time of the vote. The probability of the event that the vote totals are $k$ and $l$, written $(k, l)$, irrespective of the size of the electorate, is

$$\Pr[(k, l) | \alpha] = \sum_{m=k+l}^{\infty} \Pr[(k, l; m) | \alpha]$$

$$= e^{-n\sigma_A} \frac{(n\sigma_A)^k}{k!} e^{-n\sigma_B} \frac{(n\sigma_B)^l}{l!}$$

The probability $\Pr[(k, l) | \beta]$ of the event $(k, l)$ in state $\beta$ may similarly be obtained by replacing $\sigma$ with $\tau$.

Under majority rule, the probability of the event $T$ that there is a tie in state $\alpha$ is

$$\Pr[T | \alpha] = e^{-n(\sigma_A + \sigma_B)} \sum_{k=0}^{\infty} \frac{(n\sigma_A)^k}{k!} \frac{(n\sigma_B)^k}{k!}$$

(1)

The probability of the event $T_{-1}$ that $A$ is one vote short of a tie state $\alpha$ is

$$\Pr[T_{-1} | \alpha] = e^{-n(\sigma_A + \sigma_B)} \sum_{k=1}^{\infty} \frac{(n\sigma_A)^{k-1}}{(k-1)!} \frac{(n\sigma_B)^k}{k!}$$

(2)

The probability of the event $T_{+1}$ that $A$ has one more vote than that needed for a tie (or equivalently, that $B$ is one vote short of a tie) may be written by exchanging $\sigma_A$ and $\sigma_B$ in (2).

The corresponding probabilities in state $\beta$ are obtained by substituting $\tau$ for $\sigma$.
2.2.1 Asymptotic Formulae

Consider the following functions:

\[ I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k! k!} \]

\[ I_1(z) = \sum_{k=1}^{\infty} \frac{\left(\frac{z}{2}\right)^{k-1}}{(k-1)! k!} \]

\( I_0 \) and \( I_1 \) are called modified Bessel functions of order 0 and 1, respectively (see Abramowitz and Stegun, 1965). In terms of modified Bessel functions, we can rewrite (1), (2), etc. as

\[
\Pr [T \mid \alpha] = e^{-n(\sigma_A + \sigma_B)} I_0 \left(2n\sqrt{\sigma_A \sigma_B}\right)
\]

\[
\Pr [T_{\pm 1} \mid \alpha] = e^{-n(\sigma_A + \sigma_B)} \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm \frac{1}{2}} I_1 \left(2n\sqrt{\sigma_A \sigma_B}\right)
\]

(3)

Again, the corresponding probabilities in state \( \beta \) are found by substituting \( \tau \) for \( \sigma \).

It is well known (Abramowitz and Stegun, 1965) that when \( z \) is large

\[
I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}} \approx I_1(z)
\]

(4)

The approximation in (4) implies that if \( n\sqrt{\sigma_A \sigma_B} \to \infty \), as \( n \to \infty \) then, for large \( n \)

\[
\Pr [T \mid \alpha] \approx \frac{e^{-n(\sigma_A + \sigma_B - 2\sqrt{\sigma_A \sigma_B})}}{\sqrt{4\pi n\sqrt{\sigma_A \sigma_B}}} = \frac{e^{-n(\sqrt{\sigma_A} - \sqrt{\sigma_B})^2}}{\sqrt{4\pi n\sqrt{\sigma_A \sigma_B}}}
\]

(5)

Also, the probability of “offset” events of the form \( T_{+1} \) or \( T_{-1} \) can be approximated as follows

\[
\Pr [T_{\pm 1} \mid \alpha] \approx \Pr [T \mid \alpha] \times \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm \frac{1}{2}}
\]

(6)

And of course, the corresponding probabilities in state \( \beta \) can again be approximated by substituting \( \tau \) for \( \sigma \).

The probabilities of the pivotal events defined in Section 2.1 can then be approximated by using (5) and (6). In state \( \alpha \),

\[
\Pr [Piv_A \mid \alpha] \approx \frac{1}{2} \Pr [T \mid \alpha] \times \left(1 + \sqrt{\frac{\sigma_B}{\sigma_A}}\right)
\]

(7)

\[
\Pr [Piv_B \mid \alpha] \approx \frac{1}{2} \Pr [T \mid \alpha] \times \left(1 + \sqrt{\frac{\sigma_A}{\sigma_B}}\right)
\]

(8)

Again, the pivot probabilities in state \( \beta \) can similarly be obtained by substituting \( \tau \) for \( \sigma \) so that

\[
\Pr [Piv_A \mid \beta] \approx \frac{1}{2} \Pr [T \mid \beta] \times \left(1 + \sqrt{\frac{\sigma_B}{\tau_A}}\right)
\]

(9)

\[
\Pr [Piv_B \mid \beta] \approx \frac{1}{2} \Pr [T \mid \beta] \times \left(1 + \sqrt{\frac{\tau_A}{\sigma_B}}\right)
\]

(10)

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\[ ^1 \text{X}(n) \approx Y(n) \text{ means that } \lim_{n \to \infty} \left(\frac{X(n)}{Y(n)}\right) = 1. \]
The approximation formulae for the pivot probabilities also follow from Myerson (2000).

3 Compulsory Voting

Our main concern in this paper is with situations in which voting is voluntary. As a benchmark, however, we begin by considering situations in which no abstentions are allowed—all citizens must vote for either A or B. As first shown by Austen-Smith and Banks (1996), it is then typically the case that sincere voting does not constitute an equilibrium.

To see why this is also true in the context of our model, suppose that all voters vote sincerely; that is, all those with a signal of \(a\) vote for A and all those with a signal of \(b\) vote for B. For this to constitute an equilibrium, it must be the case that, after receiving signal \(a\), a voter’s expected payoff from voting for A exceeds her payoff from voting for B:

\[
q(\alpha | a) \Pr[Piv_A | a] - q(\beta | a) \Pr[Piv_A | \beta] 
\]

Rearranging terms, the “incentive compatibility” constraint for “type \(a\)” voters becomes

\[
q(\alpha | a) (\Pr[Piv_A | \alpha] + \Pr[Piv_B | \alpha]) 
\geq q(\beta | a) (\Pr[Piv_A | \beta] + \Pr[Piv_B | \beta])
\]

Because voting is compulsory and all citizens vote sincerely, in state \(\alpha\) the probability that a vote is cast for A is the same as the probability \(r\) that an \(a\) signal is received. Thus, \(\sigma_A = r\) and since there are no abstentions, \(\sigma_B = 1 - r\). Similarly in state \(\beta\) the probability that a vote is cast for A is the same as the probability \(1 - s\) that an \(a\) signal is received. So \(\tau_A = 1 - s\) and \(\tau_B = s\).

Since both \(n\sigma \to \infty\) and \(n\tau \to \infty\), the Myerson approximations of the pivot probabilities imply that for large \(n\), the ratio

\[
\frac{\Pr[Piv_A | \alpha] + \Pr[Piv_B | \alpha]}{\Pr[Piv_A | \beta] + \Pr[Piv_B | \beta]} 
\approx \frac{\Pr[T | \alpha]}{\Pr[T | \beta]} \times K(r, s)
\]

\[
\approx e^{2n(\sqrt{r(1-r)} - \sqrt{s(1-s)})} \times K(r, s)
\]

where \(K(r, s)\) is a ratio of the offset terms in (7) and (8) and the corresponding offset terms in (9) and (10). It is easy to see that \(K(r, s)\) is positive and, with compulsory voting, independent of \(n\).

If \(r > s > \frac{1}{2}, s(1-s) > r(1-r)\) and so the expression in (11) goes to zero as \(n\) increases. This implies that, when \(n\) is large and a voter is pivotal, state \(\beta\) is infinitely more likely than is state \(\alpha\). Thus, the incentive compatibility condition for those with signal \(a\), given in ICa, cannot hold. Hence a voter with a \(a\) signal will not wish to vote sincerely in a large election.\(^2\)

\(^2\)If \(r = s\), then the ratio of the pivot probabilities is always 1 and incentive compatibility holds.
**Proposition 1** If voting is compulsory, sincere voting is not an equilibrium in large elections with different signal precisions ($r \neq s$).

In Section 7 below, we reexamine compulsory voting in more detail with a view to comparing it to the case of voluntary voting.

4 Voluntary Voting

In this section, we simultaneously introduce two features to the model. First, we allow for the possibility of abstention—every citizen need not vote. Second, we suppose that voters incur some costs of going to the polls to cast their votes and that these costs can be avoided by staying at home.

We suppose that voting costs vary across voters. Specifically, the cost of voting for each voter is private information and determined by a realization from a continuous probability distribution $F$ with support $[0, 1]$. We suppose that $F$ admits a density $f$ that is strictly positive on $(0, 1)$. Finally, we assume that voting costs are independently distributed across voters and independent of the signal as to who is the better candidate.

Thus prior to the voting decision, each voter has two pieces of private information—his cost of voting and a signal regarding the state. We will show that when $n$ is large, there exists a unique equilibrium of the voting game with the following features.

1. There exists a pair of positive threshold costs, $c_a$ and $c_b$, such that a citizen with a cost realization $c$ and who receives a signal $i = a, b$ votes if and only if $c \leq c_i$. The threshold costs determine differential participation rates $F(c_a) = p_a$ and $F(c_b) = p_b$.

2. All those who vote do so sincerely—that is, all those with a signal of $a$ vote for $A$ and those with a signal of $b$ vote for $B$.

To summarize, in the model with voluntary and costly voting, our main result is

**Theorem 1** In large elections with voluntary voting, there is a unique equilibrium. In this equilibrium, (i) all voters vote sincerely; (ii) the right candidate is elected with probability one.

The result is established in four steps.

1. First, we consider only the participation decision. Under the assumption of sincere voting, we establish the existence of positive threshold costs and the corresponding participation rates. This step does not require the assumption that the expected size of the electorate is large.

This corresponds to one of the non-generic cases identified by Austen-Smith and Banks (1996) in a fixed $n$ model. See also Myerson (1998).
2. Second, we show that with a large electorate, given the participation rates determined in Step 1, it is indeed an equilibrium to vote sincerely.

3. Third, we show that the participation rates are such that, in the limit, information fully aggregates—the right candidate is chosen with probability one.

4. Fourth, we show that with a large electorate, the equilibrium is unique.

### 4.1 Equilibrium Participation Rates

We now show that when all those who vote do so sincerely, there is an equilibrium in cutoff strategies. That is, there exists a threshold cost \( c_a > 0 \) such that all voters receiving a signal of \( a \) and having a cost \( c \leq c_a \) go to the polls and vote for \( A \). Analogously, there exists a threshold cost \( c_b > 0 \) for voters with a signal of \( b \). Equivalently, one can think of a participation probability, \( p_a = F(c_a) \) that a voter with an \( a \) signal goes to the polls and a probability \( p_b = F(c_b) \) that a voter with a \( b \) signal goes to the polls.

Under these conditions, a given voter will vote for \( A \) in state \( \alpha \) only if he receives the signal \( a \) (which happens with probability \( r \)) and has a voting cost lower than \( c_a \) (which happens with probability \( p_a \)). Thus the probability of a vote for \( A \) in state \( \alpha \) is \( \sigma_A = rp_a \). Similarly, the probability of a vote for \( B \) in state \( \alpha \) is \( \sigma_B = (1 - r)p_b \). The probabilities of voting for \( A \) and \( B \) in state \( \beta \) are \( \tau_A = (1 - s)p_a \) and \( \tau_B = sp_b \), respectively.

We look for participation rates \( p_a \) and \( p_b \) such that a voter with signal \( a \) and cost \( c_a = F^{-1}(p_a) \) is indifferent between going to the polls and staying home. Formally, this amounts to the condition that

\[
U_a (p_a, p_b) \equiv q(\alpha | a) \Pr[Piv_A | \alpha] - q(\beta | a) \Pr[Piv_A | \beta] = F^{-1}(p_a) \quad (\text{IRa})
\]

where the pivot probabilities are determined using the voting probabilities \( \sigma \) and \( \tau \) as above. Likewise, a voter with signal \( b \) cost \( c_b = F^{-1}(p_b) \) must also be indifferent.

\[
U_b (p_a, p_b) \equiv q(\beta | b) \Pr[Piv_B | \beta] - q(\alpha | b) \Pr[Piv_B | \alpha] = F^{-1}(p_b) \quad (\text{IRb})
\]

**Proposition 2** There exist participation rates \( p_a^* \in (0, 1) \) and \( p_b^* \in (0, 1) \) that simultaneously satisfy IRa and IRb.

**Proof.** It is useful to rewrite IRa and IRb in terms of threshold costs rather than participation probabilities. Let \( V_a (c_a, c_b) \) denote the payoff to a voter with signal \( a \) from voting for \( A \) when the two threshold costs are \( c_a = F^{-1}(p_a) \) and \( c_b = F^{-1}(p_b) \); that is, \( V_a (c_a, c_b) \equiv U_a (F(c_a), F(c_b)) \). Similarly, let \( V_b (c_a, c_b) \equiv U_b (F(c_a), F(c_b)) \). We will show that there exist \( (c_a, c_b) \in (0,1)^2 \) such that \( V_a (c_a, c_b) = c_a \) and \( V_b (c_a, c_b) = c_b \).

The function \( V = (V_a, V_b) : [0,1]^2 \rightarrow [-1,1]^2 \) maps a pair of threshold costs to a pair of payoffs from voting sincerely. Note that payoffs may be negative.

Consider the function \( V^+ : [0,1]^2 \rightarrow [0,1]^2 \) defined by

\[
V_a^+ (c_a, c_b) = \max \{0, V_a (c_a, c_b)\}
\]

\[
V_b^+ (c_a, c_b) = \max \{0, V_b (c_a, c_b)\}
\]
Since $V$ is a continuous function, $V^+$ is also continuous and so by Brouwer’s Theorem $V^+$ has a fixed point, say $(c_a^*, c_b^*) \in [0, 1]^2$.

We argue that $c_a^*$ and $c_b^*$ are strictly positive. Suppose that $c_a^* = 0$. Then $p_a^* = F(c_a^*)$ is also zero and so there are no $a$ types who vote. Consider an individual who receives a signal of $a$. The only events in which a vote for $A$ is pivotal is if either (i) no $b$ types show up to vote; or (ii) a single $b$ type shows up. Thus

$$Pr[Piv_A | \alpha] = \frac{1}{2}e^{-n(1-r)p_b^*} \left( 1 + n(1-r)p_b^* \right)$$
$$Pr[Piv_A | \beta] = \frac{1}{2}e^{-np_b^*} \left( 1 + np_b^* \right)$$

where $p_b^* = F(c_b^*)$. We claim that $Pr[Piv_A | \alpha] > Pr[Piv_A | \beta]$. This follows from the fact that the function $g(x) = e^{-x}(1+x)$ is strictly decreasing for $x > 0$ and that $s > 1 - r$. Hence, if $p_a^* = 0$

$$q(\alpha | a) Pr[Piv_A | \alpha] - q(\beta | a) Pr[Piv_A | \beta] > 0$$

since $q(\alpha | a) > \frac{1}{2}$. Since $c_a^* = 0$, this is equivalent to

$$V_a^+(c_a^*, c_b^*) > c_a^*$$

contradicting the assumption that $(c_a^*, c_b^*)$ was a fixed point. Thus $c_a^* > 0$.

A similar argument shows that $c_b^* > 0$.

Since both $c_a^*$ and $c_b^*$ are strictly positive, we have that

$$V^+(c_a^*, c_b^*) = V(c_a^*, c_b^*) = (c_a^*, c_b^*)$$

Thus $(c_a^*, c_b^*)$ is also a fixed point of $V$ and so solves $IRa$ and $IRb$.

Next, notice that at any point $(1, p_b)$

$$q(\alpha | a) Pr[Piv_A | \alpha] - q(\beta | a) Pr[Piv_A | \beta] < 1$$

Thus if $(c_a^*, c_b^*)$ is a fixed point of $V$ then we also have that both $c_a^*$ and $c_b^*$ are also less than one. ■

**Lemma 1** If $r > s$, then any solution to $IRa$ and $IRb$ satisfies $p_a^* < p_b^*$.

**Proof.** We will show that if $p_a \geq p_b$, then $U_a(p_a, p_b) < U_b(p_a, p_b)$. A rearrangement of the relevant terms shows that this is equivalent to

$$(q(\alpha | a) + q(\alpha | b)) Pr[T | \alpha] + q(\alpha | a) Pr[T_{-1} | \alpha] + q(\alpha | b) Pr[T_{+1} | \alpha] \quad (12)$$

being less than

$$(q(\beta | b) + q(\beta | a)) Pr[T | \beta] + q(\beta | a) Pr[T_{-1} | \beta] + q(\beta | b) Pr[T_{+1} | \beta] \quad (13)$$

With sincere voting, $\sigma_A = rp_a$, $\sigma_B = (1-r)p_b$, $\tau_A = (1-s)p_a$ and $\tau_B = sp_b$. 

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First, since \( r > s > \frac{1}{2} \), we have \( \sigma_A \sigma_B < \tau_A \tau_B \) and since \( p_a \geq p_b \), \( \sigma_A + \sigma_B \geq \tau_A + \tau_B \). Thus, 

\[
Pr[T | \alpha] = e^{-n(\sigma_A + \sigma_B)} \sum_{k=0}^{\infty} \frac{(n\sigma_A)^k}{k!} \frac{(n\sigma_B)^k}{k!} < e^{-n(\tau_A + \tau_B)} \sum_{k=0}^{\infty} \frac{(n\tau_A)^k}{k!} \frac{(n\tau_B)^k}{k!} = Pr[T | \beta]
\]

Second, since \( r > s > \frac{1}{2} \), we have for all \( k \geq 1 \), \( r (\sigma_A)^{k-1} (\sigma_B)^{k} < (1 - s) (\tau_A)^{k-1} (\tau_B)^{k} \). Thus, 

\[
q(\alpha | a) Pr[T_{-1} | \alpha] = e^{-n(\sigma_A + \sigma_B)} \frac{r}{r + 1 - s} \sum_{k=1}^{\infty} \frac{(n\sigma_A)^{k-1}}{(k-1)!} \frac{(n\sigma_B)^k}{k!} < e^{-n(\tau_A + \tau_B)} \frac{1 - s}{r + 1 - s} \sum_{k=1}^{\infty} \frac{(n\tau_A)^{k-1}}{(k-1)!} \frac{(n\tau_B)^k}{k!} = q(\beta | a) Pr[T_{-1} | \beta]
\]

A similar argument establishes that 

\[q(\alpha | b) Pr[T_{+1} | \alpha] < q(\beta | b) Pr[T_{+1} | \beta]\]

Combining these with the fact that \( q(\alpha | a) < q(\beta | b) \) establishes that (12) is less than (13).

This means that if \( p_a^* \geq p_b^* \), then \((p_a^*, p_b^*)\) cannot satisfy IRa and IRb. Thus \( p_a^* < p_b^* \).

The workings of the proposition may be seen in the following example.

**Example 1** Consider an expected electorate \( n = 100 \). Suppose the signal precisions \( r = \frac{3}{4} \) and \( s = \frac{2}{3} \) and that the voting costs distributed according to \( F(c) = c^\frac{1}{3} \). Then \( p_a^* = 0.152 \) and \( p_b^* = 0.181 \).

Figure 1 depicts the IRa and IRb curves for this example. Notice that neither curve defines a function and that IRa is in two pieces.

### 4.2 Participation Rates in Large Elections

In Proposition 2 we showed that under the assumption of sincere voting, for all \( n \), there exist a pair of positive equilibrium participation rates. In this section, we study the limiting behavior of these rates. We will show that although the participation rates go to zero as \( n \) increases, they do so sufficiently slowly so that the expected number of voters goes to infinity.

As a first step we have\(^3\) 

\(^3\)Unless otherwise specified, all limits are taken as \( n \to \infty \).
Lemma 2  In any sequence of sincere voting equilibria, the threshold costs tend to zero; that is, \( \limsup c_a(n) = \limsup c_b(n) = 0 \).

Proof. Suppose to the contrary, that for some sequence, \( \lim c_a(n) > 0 \). In that case, the gross benefits (excluding the costs of voting) to voters with \( a \) signals from voting must be positive; that is

\[
\lim (q(\alpha | a) \Pr [Pi\nu_A | \alpha] - q(\beta | a) \Pr [Pi\nu_A | \beta]) > 0
\]

where it is understood that the probabilities depend on \( n \).

We know that along the given sequence, \( \lim p_a(n) > 0 \). This implies that \( \lim \sigma_A(n) = \lim rp_a(n) > 0 \) also and hence \( \lim (n\sigma_A) = \infty \).

First, suppose that there is a subsequence along which \( \lim (n\sqrt{\sigma_A\sigma_B}) < \infty \). In that case,

\[
\Pr [Pi\nu_A | \alpha] = e^{-n(\sigma_A+\sigma_B)} \left( I_0 \left( 2n\sqrt{\sigma_A\sigma_B} \right) + \frac{n\sigma_B}{n\sigma_A} I_1 \left( 2n\sqrt{\sigma_A\sigma_B} \right) \right)
\]

and since \( (e^{-n\sigma_A}/\sqrt{n\sigma_A}) = 0 \) and \( \limsup e^{-n\sigma_B}\sqrt{n\sigma_B} < \infty \), along any such subsequence,

\[
\lim \Pr [Pi\nu_A | \alpha] = 0
\]
Second, suppose that there is a subsequence along which \( \lim (n\sqrt{\sigma_A \sigma_B}) = \infty \). In that case,

\[
\Pr[\text{Piv}_A \mid \alpha] \approx \frac{e^{-n(\sigma_A + \sigma_B - 2\sqrt{\sigma_A \sigma_B})}}{\sqrt{4\pi n \sigma_A \sigma_B}} \left( 1 + \sqrt{\frac{2\sigma_B}{\sigma_A}} \right)
\]

Notice that the denominator is unbounded while the numerator is always bounded. Hence, along any such subsequence,

\[
\lim \Pr[\text{Piv}_A \mid \alpha] = 0
\]

An identical argument applies for \( \tau_A(n) \) and \( \tau_B(n) \). Therefore,

\[
\lim \Pr[\text{Piv}_A \mid \beta] = 0
\]

But this means that the gross benefit of voting for \( A \) when the signal is \( a \) tends to zero. This contradicts the assumption that \( \lim c_n(n) > 0 \).

Next, we show that, despite the fact that the threshold cost for voting goes to zero in the limit, in any sincere voting equilibrium, the expected number of voters with signal \( a \) or \( b \) is unbounded in large elections. There is a “race” between the speed at which the participation thresholds approach zero relative to the size of the electorate. A common intuition one may have is that the “winner” of this race depends on the shape of the cost distribution—particularly in the neighborhood of 0. As we show below, however, sincere voting equilibria have the property that, in large elections, the number of voters becomes unbounded regardless of the shape of this distribution. That is, in the sincere voting model, the problem of too little participation to achieve information aggregation does not arise in the limit.

**Lemma 3** Suppose that there is a sequence of sincere voting equilibria such that \( \lim n p_a(n) = n_a < \infty \) and \( \lim n p_b(n) = n_b < \infty \). If \( r n_a \geq (1 - r) n_b \), then \( U_b = 0 \) implies \( U_a > 0 \).

**Proof.** The condition that \( U_b = 0 \) is equivalent to

\[
s \Pr[\text{Piv}_B \mid \beta] = (1 - r) \Pr[\text{Piv}_B \mid \alpha]
\]

whereas \( U_a > 0 \) is equivalent to

\[
r \Pr[\text{Piv}_A \mid \alpha] > (1 - s) \Pr[\text{Piv}_A \mid \beta]
\]

We will argue that

\[
\frac{r \Pr[\text{Piv}_A \mid \alpha]}{(1 - s) \Pr[\text{Piv}_A \mid \beta]} > \frac{(1 - r) \Pr[\text{Piv}_B \mid \alpha]}{s \Pr[\text{Piv}_B \mid \beta]}
\]

or equivalently,

\[
\frac{r n_a (\Pr[T \mid \alpha] + \Pr[T_{-1} \mid \alpha])}{(1 - s) n_a (\Pr[T \mid \beta] + \Pr[T_{-1} \mid \beta])} > \frac{(1 - r) n_b (\Pr[T \mid \alpha] + \Pr[T_{+1} \mid \alpha])}{s n_b (\Pr[T \mid \beta] + \Pr[T_{+1} \mid \beta])}
\]
Now note that
\[ rn_a \Pr[T_{-1} | \alpha] = (1 - r) n_b \Pr[T_{+1} | \alpha] \]
and
\[ (1 - s) n_a \Pr[T_{-1} | \beta] = s n_b \Pr[T_{+1} | \beta] \]
and the required inequality follows from the fact that \( rn_a \geq (1 - r) n_b \) and \( (1 - s) n_a < s n_b \).

Lemma 4 Suppose that there is a sequence of sincere voting equilibria such that \( \lim np_a (n) = n_a < \infty \) and \( \lim np_b (n) = n_b < \infty \). If \( rn_a < (1 - r) n_b \), then \( U_a > 0 \).

Proof. Consider the function
\[ G(x, y) = e^{-x-y} (x I_0(z) + \frac{1}{2} z I_1(z)) \]
where \( z = 2\sqrt{xy} \).

Note that if we write \( rn_a = x_\alpha \) and \( (1 - r) n_a = y_\alpha \), then
\[ rn_a \Pr[Piv_A | \alpha] = x_\alpha e^{-x_\alpha-y_\alpha} \left( I_0(2\sqrt{x_\alpha y_\alpha}) + \sqrt{\frac{y_\alpha}{x_\alpha}} I_1(2\sqrt{x_\alpha y_\alpha}) \right) = \frac{1}{2} G(x_\alpha, y_\alpha) \]
Similarly, if we write \( (1 - s) n_a = x_\beta \) and \( s n_\beta = y_\beta \), then
\[ (1 - s) n_a \Pr[Piv_A | \beta] = \frac{1}{2} G(x_\beta, y_\beta) \]
We will show that when \( x < y \), \( G(x, y) \) is increasing in \( x \) and decreasing in \( y \).
Observe that
\[ G_x(x, y) = e^{-x-y} \left( I_0'(z) + x I_0'(z) z_x + \frac{1}{2} (z I_1(z))' z_x - x I_0(z) - \frac{1}{2} z I_1(z) \right) \]
\[ = e^{-x-y} \left( I_0'(z) + x I_1(z) z_x + \frac{1}{2} z I_0(z) z_x - x I_0(z) - \frac{1}{2} z I_1(z) \right) \]
\[ = e^{-x-y} (1 + y - x) I_0(z) > 0 \]
where we have used the fact that \( I_0'(z) = I_1(z) \) and \( (z I_1(z))' = z I_0(z) \). Also, \( xz_x = \frac{1}{2} z \) and \( \frac{1}{2} z z_x = y \).

Also,
\[ G_y(x, y) = e^{-x-y} \left( x I_0'(z) z_y + \frac{1}{2} (z I_1(z))' z_y - x I_0(z) - \frac{1}{2} z I_1(z) \right) \]
\[ = e^{-x-y} \left( x I_1(z) z_y + \frac{1}{2} z I_0(z) z_y - x I_0(z) - \frac{1}{2} z I_1(z) \right) \]
\[ = e^{-x-y} (x z_y - \frac{1}{2} z) I_1(z) < 0 \]
where we have used the fact that, \( z_y z = 2x \) and \( z_y = \sqrt{\frac{2}{y}} < 1 \) and \( x < \frac{1}{2} z \).
Finally, notice that since \( r n_a < (1 - r) n_b \)
\[(1 - s) n_a < r n_a < (1 - r) n_b < s n_b \]
which is the same as
\[ x_\beta < x_\alpha < y_\alpha < y_\beta \]
and since \( G_x > 0 \) and \( G_y < 0 \) for \( x < y \), we have
\[ \frac{r \Pr [Piv_A \mid \alpha]}{(1 - s) \Pr [Piv_A \mid \beta]} = \frac{G(x_\alpha, y_\alpha)}{G(x_\beta, y_\beta)} > 1 \]
and so
\[ U_a = r \Pr [Piv_A \mid \alpha] - (1 - s) \Pr [Piv_A \mid \beta] > 0 \]

**Lemma 5** In any sequence of sincere voting equilibria, either
\[ \lim n p_a (n) = \infty \] or
\[ \lim n p_b (n) = \infty . \]

**Proof.** Lemma 2 then implies that both
\[ \lim U_a (p_a (n), p_b (n)) = 0 \] and
\[ \lim U_b (p_a (n), p_b (n)) = 0 \]
Suppose to the contrary that \( \lim n p_a (n) < \infty \) and \( \lim n p_b (n) < \infty . \) But now Lemmas 3 and 4 lead to a contradiction.

Our next lemma shows that in the limit, the participation rates are of the same order of magnitude.

**Lemma 6** In any sequence of sincere voting equilibria, (i) \[ \lim \inf \frac{p_a (n)}{p_b (n)} > 0 ; \] and (ii) \[ \lim \inf \frac{p_b (n)}{p_a (n)} > 0 . \]

**Proof.** To prove part (i), suppose to the contrary that \( \lim \inf \frac{p_a (n)}{p_b (n)} = 0 . \) Lemma 5 implies that \( \lim \inf \frac{n p_b (n)}{p_a (n)} = \infty . \)
Consider the probability of outcome \( (k, l) \) in state \( \alpha \)
\[ \Pr [(k, l) \mid \alpha] = e^{-n r p_a} \frac{(n r p_a)^k}{k!} e^{-n (1 - r) p_b} \frac{(n (1 - r) p_b)^l}{l!} \]
and the corresponding probability \( \Pr [(k, l) \mid \beta] \), which is obtained by substituting
\( (1 - s) \) for \( r \) in the expression above.

The likelihood ratio
\[ \frac{\Pr [(k, l) \mid \alpha]}{\Pr [(k, l) \mid \beta]} = e^{n p_b (r + s - 1) (1 - p_b / p_a)} \frac{r}{(1 - s)^k} \frac{(1 - r)^l}{s} \]
Since along some sequence, \( \frac{p_a}{p_b} \to 0 \) and \( n p_b \to \infty \)
\[ e^{n p_b (r + s - 1) (1 - p_a / p_b)} \to \infty \]
Moreover, in all events in the set $Piv_B$, $|k - l| \leq 1$.

Thus, there exists an $n_0$ such that for all $n \geq n_0$

$$\frac{\Pr[Piv_B | \alpha]}{\Pr[Piv_B | \beta]} > q (\beta | b) / q (\alpha | b)$$

But this contradicts the fact that for all $n$, the participation thresholds are positive, that is

$$q (\beta | b) \Pr[Piv_B | \beta] - q (\alpha | b) \Pr[Piv_B | \alpha] = F^{-1} (p_b) > 0$$

Part (ii) is, of course, an immediate consequence of Lemma 1.

We are now in a position to show

**Proposition 3** In any sequence of sincere voting equilibria, the expected number of voters with either signal tends to infinity; that is,

$$\lim \inf n p_a (n) = \infty = \lim \inf n p_b (n)$$

**Proof.** The proof is a direct consequence of Lemmas 5 and 6.

To summarize, we have shown that, even though the cost thresholds for participation go to zero, they do so sufficiently slowly that the expected number of voters with $a$ and with $b$ signals is unbounded as $n$ gets arbitrarily large.

### 4.3 Sincere Voting

In this subsection we establish that given the participation rates as determined above, it is a best-response for every voter to vote sincerely.

We have argued that any sequence of equilibrium participation rates has the property that $n p_a (n) \rightarrow \infty$ and $n p_b (n) \rightarrow \infty$. The following result is key—it compares the likelihood ratio of $\alpha$ to $\beta$ conditional on the event $Piv_B$ to that conditional on the event $Piv_A$.

**Lemma 7 (Likelihood Ratio)** Let $(p_a (n), p_b (n))$ be a sequence of participation rates such that $n \sqrt{p_a (n)} p_b (n) \rightarrow \infty$. Then, for large $n$,

$$\frac{\Pr[Piv_B | \alpha]}{\Pr[Piv_B | \beta]} \bigg/ \frac{\Pr[Piv_A | \alpha]}{\Pr[Piv_A | \beta]}$$

**Proof.** When $n$ is large, the approximation formulae (7) to (10) are valid for $\sigma_A = r p_a$, $\sigma_B = (1 - r) p_b$, $\tau_A = (1 - s) p_a$ and $\tau_B = s p_b$. Using these formulae, the required inequality reduces to

$$\frac{1 + \sqrt{\frac{sp_b}{(1-s)p_a}}}{1 + \sqrt{\frac{(1-r)p_b}{r p_a}}} > \frac{1 + \sqrt{\frac{(1-s)p_a}{sp_b}}}{1 + \sqrt{\frac{r p_b}{(1-r)p_a}}}$$
and a term-by-term comparison shows that this clearly holds since both \( r \) and \( s \) are greater than \( \frac{1}{2} \). ■

We finally turn to incentive compatibility—each citizen who votes is better off voting according to his signal than otherwise.

For some large \( n \), let \((p_a, p_b)\) be the equilibrium participation rates. At these participation rates, a voter with signal \( a \) and cost \( c_a = F^{-1}(p_a) \) is just indifferent between voting and staying home, that is,

\[
q(\alpha | a) \Pr[Piv_A | \alpha] - q(\beta | a) \Pr[Piv_A | \beta] = F^{-1}(p_a)
\]

We want to show that the incentive compatibility constraint for “type \( a \)” voters is satisfied, that is,

\[
q(\alpha | a) (\Pr[Piv_A | \alpha] + \Pr[Piv_B | \alpha]) \geq q(\beta | a) (\Pr[Piv_A | \beta] + \Pr[Piv_B | \beta])
\]

Given IRA, however, ICa is implied by

\[
q(\alpha | a) \Pr[Piv_B | \alpha] - q(\beta | a) \Pr[Piv_B | \beta] > 0
\]

Now notice that since \( p_a > 0 \), IRA implies

\[
\frac{\Pr[Piv_A | \alpha]}{\Pr[Piv_A | \beta]} > \frac{q(\beta | a)}{q(\alpha | a)}
\]

and so applying Lemma 7 we get that for large \( n \),

\[
\frac{\Pr[Piv_B | \alpha]}{\Pr[Piv_B | \beta]} > \frac{q(\beta | a)}{q(\alpha | a)}
\]

which is the same as (14).

Thus we have argued that if \((p_a, p_b)\) are such that a voter with signal \( a \) and cost \( F^{-1}(p_a) \) is just indifferent between participating or not, then all voters with \( a \) signals who have lower costs, have the incentive to vote for \( A \) over \( B \). In other words, all those with signal \( a \) who do vote should vote sincerely. Recall that under compulsory voting, it was not in the interests of those with \( a \) signals to vote sincerely.

What about voters with \( b \) signals? Again, since \((p_a, p_b)\) are equilibrium participation rates, then a voter with signal \( b \) and cost \( c_b = F^{-1}(p_b) \) is just indifferent between voting and staying home, that is,

\[
q(\beta | b) \Pr[Piv_B | \beta] - q(\alpha | b) \Pr[Piv_B | \alpha] = F^{-1}(p_b)
\]

We want to show that a voter with signal \( b \) is better off voting for \( B \) over \( A \), that is

\[
q(\beta | b) (\Pr[Piv_A | \beta] + \Pr[Piv_B | \beta]) \geq q(\alpha | b) (\Pr[Piv_A | \alpha] + \Pr[Piv_B | \alpha])
\]

Again, since \( p_b > 0 \), the left-hand side of IRb is strictly positive and so ICb is implied by

\[
q(\beta | b) \Pr[Piv_A | \beta] - q(\alpha | b) \Pr[Piv_A | \alpha] > 0
\]

As above, for large \( n \), this is also implied by Lemma 7.

We have thus established
Figure 2: Equilibrium with Sincere Voting

**Proposition 4** If participation rates are determined by IRa and IRb, then for large $n$, sincere voting is incentive compatible.

In the proof of the proposition above we used the assumption that the expected number of voters is large. This enabled us to exploit the approximation formulae for the pivot probabilities derived by Myerson (2000). As the following example shows, however, a sincere voting equilibrium may exist even when $n$ is small (in fact, we know of no example in which such an equilibrium does not exist).

**Example 2** Suppose that the expected size of the electorate $n = 10$, the signal precisions $r = \frac{3}{4}$, $s = \frac{2}{3}$, and cost distribution $F(c) = \sqrt{2c}$ on $[0, \frac{1}{2}]$. Then the (unique) sincere voting equilibrium has participation probabilities $p_a^* = 0.268$ and $p_b^* = 0.312$.

The relevant incentive and participation constraints for each type of voter are shown in Figure 2.

5 Information Aggregation

We now turn to the question of whether the equilibrium is efficient under costly voting. In other words, is it the case that in large elections, the “right” candidate is
elected? One may have thought that we have, in effect, already answered this question (in the affirmative) by showing that voting is sincere and expected participation is unbounded in large elections. However, this ignores that the fact that voters with different signals turn out at different rates. If turnout is too lop-sided in favor of B versus A, then even with sincere voting, the election could still fail to choose the “right” candidate.

In large elections, candidate A is chosen in state $\alpha$ if and only if
\[ rp_a > (1 - r) p_b \]
and candidate B is chosen in state $\beta$ if and only if
\[ (1 - s) p_a < sp_b \]
Information aggregation thus requires that for large $n$, the ratio of the equilibrium participation rates satisfies
\[ \frac{1 - r}{r} < \frac{p_a}{p_b} < \frac{s}{1 - s} \]
First, recall from Lemma 1 that any solution to the threshold equations satisfies $p_a < p_b$. Thus in large elections, in equilibrium, $b$ types turn out to vote at higher rates than do $a$ types. Since $s > \frac{1}{2}$, this implies that the second inequality holds and so in large elections, $B$ wins in state $\beta$ with probability 1.

In state $\alpha$, however, the larger turnout for $B$ is detrimental. We now argue that in large elections, the first inequality also holds.

First, note from Lemma 6 that since with sincere voting
\[ \frac{\sigma_A}{\sigma_B} = \frac{rp_a}{(1 - r) p_b} \quad \text{and} \quad \frac{\tau_A}{\tau_B} = \frac{(1 - s) p_a}{sp_b} \]
it is the case that
\[ \limsup \left( \frac{\sigma_A}{\sigma_B} \right)^{\frac{1}{2}} < \infty \quad \text{and} \quad \limsup \left( \frac{\tau_A}{\tau_B} \right)^{\frac{1}{2}} < \infty \]
and hence, in the expressions for the pivot probabilities (see (7) to (10)), the exponential terms dominate in the limit. Thus we have
\[ \frac{\Pr[Piv_A | \alpha]}{\Pr[Piv_A | \beta]} = \frac{e^{-n(\sqrt{\sigma_A} - \sqrt{\sigma_B})^2}}{e^{-n(\sqrt{\tau_A} - \sqrt{\tau_B})^2}} \times K(\sigma_A, \sigma_B, \tau_A, \tau_B) \]
where $K$ is a function that stays finite in the limit.

Thus it must be the case that in the limit
\[ (\sqrt{\sigma_A} - \sqrt{\sigma_B})^2 = (\sqrt{\tau_A} - \sqrt{\tau_B})^2 \]  
(15)
In particular, suppose that the left-hand side of (15) was greater than the right-hand side. In that case,
\[ \lim \frac{\Pr[Piv_A | \alpha]}{\Pr[Piv_A | \beta]} = 0 \]
and it would then follow that state $\beta$ is infinitely more likely in the event $Piv_A$ than is state $\alpha$. This, however, would imply that the gross benefit to a voter with signal $a$ from voting is negative, which contradicts Lemma 2. Similarly, if the left-hand side was smaller then it would then follow that state $\alpha$ is infinitely more likely in the event $Piv_B$ than is state $\beta$. This, however, would then imply that the gross benefit to a voter with signal $b$ from voting is negative, which also contradicts Lemma 2. Thus (15) must hold in the limit.

Under sincere voting $\sigma_A = rp_a$, $\sigma_B = (1 - r)p_b$, $\tau_A = (1 - s)p_a$, and $\tau_B = sp_b$, and so (15) can be rewritten as

$$\sqrt{s} - \sqrt{1 - s} \sqrt{\frac{p_a}{p_b}} \approx \pm \left( \sqrt{r} \sqrt{\frac{p_a}{p_b}} - \sqrt{1 - r} \right)$$

and the left-hand side is positive since $p_b > p_a$. Now observe that if $(1 - r)p_b \geq rp_a$, then we have

$$\sqrt{s} - \sqrt{1 - s} \sqrt{\frac{p_a}{p_b}} \approx \sqrt{1 - r} - \sqrt{r} \sqrt{\frac{p_a}{p_b}}$$

and this is impossible since both $r$ and $s$ are greater than $\frac{1}{2}$ (Lemma 6 ensures that $\frac{p_a}{p_b}$ is bounded). Thus we must have, that for large $n$,

$$(1 - r)p_b < p_a r$$

We have thus shown that information fully aggregates in large elections.

**Proposition 5** In any sequence of sincere voting equilibria, the probability that right candidate is elected in each state ($A$ in state $\alpha$ and $B$ in $\beta$) goes to one.

Note that as a result of the reasoning above, we know that

$$\sqrt{s} - \sqrt{1 - s} \sqrt{\frac{p_a}{p_b}} \approx \sqrt{r} \sqrt{\frac{p_a}{p_b}} - \sqrt{1 - r}$$

and so we obtain that ratio of the participation probabilities satisfies

$$\lim \sqrt{\frac{p_a}{p_b}} = \frac{\sqrt{1 - r} + \sqrt{s}}{\sqrt{r} + \sqrt{1 - s}}$$

(16)

6 Uniqueness

In this section, we show that with voluntary and costly voting, there is a unique equilibrium when $n$ is large. Recall that the equilibrium derived in the previous sections has the following features:

1. voting is sincere;
2. cost thresholds are determined by IRa and IRb.

To establish that there is a unique equilibrium, we first show that when $n$ is large, all equilibria must involve sincere voting. The second step is to show that for large $n$, there is a unique solution to IRa and IRb.

So we obtain,
Proposition 6 In large elections, there is a unique equilibrium.

Proof. See the Appendix. ■

7 Voluntary versus Compulsory Voting

In this section, we compare welfare under voluntary voting—it is possible to abstain—to that under compulsory voting—it is not possible to abstain. The comparison is influenced by the following trade-off. Under voluntary voting, (i) not everyone votes; but (ii) everyone who votes, does so sincerely. On the other hand, under compulsory voting, (i) everyone votes; but (ii) voters do not vote sincerely (see Proposition 1). Put another way, under voluntary voting, there is less information provided but it is accurate whereas under compulsory voting, there is more information provided but it is inaccurate. In what follows, we study this trade-off between the quality and quantity of information. The main result of this section is that in large elections, the trade-off is always resolved in favor of quality over quantity—voluntary voting is welfare superior to compulsory voting.

When comparing the two systems, we will suppose that voting costs are zero. Introducing voting costs adds an additional factor, a selection effect, which would appear to favor voluntary voting. This is because under voluntary voting only those with low realized costs turn out to vote and incur these costs whereas under compulsory voting all voters incur voting costs. Since we will show that voluntary voting is superior even when there are no costs, the ranking will obviously be unchanged if we introduce small voting costs.

7.1 Voluntary Voting with Zero Costs

Proposition 7 In large elections under voluntary voting with zero costs, there is a unique equilibrium in which (i) all $b$ types vote; (ii) all $a$ types vote with probability $p_a$; and (iii) all those who vote, vote sincerely. The sequence $p_a(n)$ satisfies

$$\lim p_a(n) = \left(\frac{\sqrt{1-r} + \sqrt{s}}{\sqrt{r} + \sqrt{1-s}}\right)^2$$ (17)

We omit a detailed proof of this proposition since this is just a limiting case of our model with voting costs. The limit of the ratio of the participation probabilities follows by setting $p_b = 1$ in equation (16).

7.2 Compulsory Voting

Proposition 8 In large elections under compulsory voting, there is a unique equilibrium in which (i) all $b$ types vote for $B$; (ii) all $a$ types vote for $A$ with probability $\mu$. The sequence $\mu(n)$ satisfies

$$\lim \mu(n) = \frac{1}{1 + r - s}$$ (18)
Again, we omit a detailed proof. The limit of the mixing probability for an \(a\) type voters may be obtained as follows. In general, we have that payoffs to an \(a\) type from voting for \(A\) and \(B\), respectively, are

\[
U(A, a) \approx e^{-n(\sqrt{\sigma_A^2 - \sigma_B^2})^2} \left( \frac{1}{2} + \sqrt{\frac{\sigma_A^2}{\sigma_A^2 - \sigma_B^2}} \right) - e^{-n(\sqrt{\tau_A^2 - \sqrt{\tau_B^2}})^2} \left( \frac{1}{2} + \sqrt{\frac{\tau_A}{\tau_A^2}} \right)
\]

\[
U(B, a) \approx e^{-n(\sqrt{\tau_B^2 - \sqrt{\tau_A^2}})^2} \left( \frac{1}{2} + \sqrt{\frac{\tau_B}{\tau_B^2}} \right) - e^{-n(\sqrt{\sigma_A^2 - \sigma_B^2})^2} \left( \frac{1}{2} + \sqrt{\frac{\sigma_B}{\sigma_B^2}} \right)
\]

where

\[
\sigma_A = r \mu, \quad \sigma_B = 1 - r \mu, \\
\tau_A = (1 - s) \mu, \quad \tau_B = 1 - (1 - s) \mu
\]

and \(\mu\) is determined by the condition that \(a\) types are indifferent between voting for \(A\) and voting for \(B\); that is, we must have

\[U(A, a) = U(B, a)\]

For this equality to hold in the limit requires that

\[2\sqrt{\sigma_A \sigma_B} - \sigma_A - \sigma_B = 2\sqrt{\tau_A \tau_B} - \tau_A - \tau_B\]

and this is easily verified to be equivalent to

\[\mu = \frac{1}{1 + r - s}\]

### 7.3 Welfare Comparison

Having derived the limiting equilibrium behavior under the two regimes—voluntary and compulsory voting with zero costs—we now use these to compare the welfare in large elections.

The social welfare \(W(\alpha)\) in state \(\alpha\) is the probability that \(A\) is elected; that is,

\[W(\alpha) = \Pr[A \text{ wins } | \alpha]\]

It will be convenient to write

\[W(\alpha) = 1 - \Pr[B \text{ wins } | \alpha] \]

\[= 1 - \frac{1}{2} \Pr[T | \alpha] - \sum_{k=1}^{\infty} \Pr[T_{-k} | \alpha]\]

where \(T_{-k}\) denotes the set of events in which \(A\) receives \(k\) fewer votes than does \(B\). Using the modified Bessel function notation we have
\[ W(\alpha) = 1 - e^{-n(\sigma_A + \sigma_B)} \left( \frac{1}{2} I_0(2n\sqrt{\sigma_A \sigma_B}) + \sum_{k=1}^{\infty} \left( \sqrt{\frac{\sigma_B}{\sigma_A}} \right)^k I_k(2n\sqrt{\sigma_A \sigma_B}) \right) \]

\[ \approx 1 - e^{-n(\sigma_A + \sigma_B)} \left( \frac{1}{2} + \sum_{k=1}^{\infty} \left( \sqrt{\frac{\sigma_A}{\sigma_B}} \right)^k \frac{e^{2n\sqrt{\sigma_A \sigma_B}}}{\sqrt{4\pi n \sqrt{\sigma_A \sigma_B}}} \right) \]

since for all \( k \), the leading term of \( I_k(z) \) is \( \frac{e^z}{\sqrt{2\pi z}} \). Thus, under the assumption that \( \sigma_B < \sigma_A \), we obtain

\[ W(\alpha) \approx 1 - e^{-n(\sqrt{\sigma_A - \sqrt{\sigma_B}})^2} \left( \frac{1}{2} + \frac{\sqrt{\frac{\sigma_B}{\sigma_A}}}{1 - \sqrt{\frac{\sigma_B}{\sigma_A}}} \right) \frac{1}{\sqrt{4\pi n \sqrt{\sigma_A \sigma_B}}} \] (19)

The welfare in state \( \beta \) can be written similarly by substituting \( \tau \) for \( \sigma \) and exchanging \( A \) and \( B \).

The main result of this section is

**Proposition 9** In large elections, the welfare in either state is higher under voluntary voting than under compulsory voting.

**Proof.** We prove that welfare in state \( \alpha \) is higher under voluntary voting than under compulsory voting. The proof for state \( \beta \) is analogous.

In the limit, under voluntary voting,

\[ \sigma_A = r\rho_A; \quad \sigma_B = 1 - r \]

whereas, in the limit, under compulsory voting

\[ \sigma_A = r\mu; \quad \sigma_B = 1 - r\mu \]

From (19), it follows that in large elections, a welfare comparison rests only on the exponential term. Specifically, we will show that the term \( \sqrt{\sigma_A} - \sqrt{\sigma_B} \) is higher under voluntary voting than under compulsory voting; that is,

\[ \sqrt{r\rho_A} - \sqrt{1 - r} > \sqrt{r\mu} - \sqrt{1 - r\mu} \] (20)

Substituting from (17) and (18) the inequality in (20) can be rewritten as

\[ \frac{\sqrt{\rho_A} + \sqrt{1 - r}}{\sqrt{\rho_A} + \sqrt{1 - s}} - \sqrt{1 - r} > \frac{\sqrt{\mu} + \sqrt{1 - r}}{\sqrt{\mu} + \sqrt{1 - s}} \]

and we will establish the inequality

\[ \sqrt{\rho_A} + \sqrt{1 - r} > \sqrt{\mu} + \sqrt{1 - s} \] (21)
which is is stronger because $r > s$, and so $1 + r - s > 1$.

The inequality in (21) may be rearranged as

$$\sqrt{r} \sqrt{s} > r + s + \sqrt{1 - r} \sqrt{1 - s} - 1$$

(22)

Now when $r = s$, the two sides are equal and it may be verified that for fixed $s$, the derivative of the left-hand side of (22) with respect to $r$ is greater than the derivative of the right-hand side. This completes the proof. □

A Appendix: Uniqueness of the Equilibrium

The purpose of this appendix is to provide a proof of Proposition 6.

First, we show that for large $n$, in any equilibrium, voting behavior must be sincere. This now means that all equilibria must be of the kind we have studied—and the only way there could be multiple equilibria is that there are multiple solutions to the equilibrium participation rates. We complete the proof by showing (in the appendix) that there can be only one pair of equilibrium participation rates.

To show that all equilibria involve sincere voting, we first rule out equilibria in which voters with $a$ signals and voters with $b$ signals both vote against their own signals with positive probability.

**Lemma 8** In any equilibrium, at least one type votes sincerely.

**Proof.** Suppose to the contrary that neither type votes sincerely.

Let $U (A, a)$ denote the gross payoff (not including costs of voting) of voting for $A$ to a voter with an $a$ signal. Similarly, define $U (B, a)$, $U (A, b)$ and $U (B, b)$.

Then we have that

$$U (A, a) > U (A, b) \geq U (B, b)$$

where the first inequality follows from the fact that all else being equal, a vote of $A$ is more valuable with signal $a$ than with with signal $b$. The second inequality follows from the fact that, by assumption, $b$ types vote for $A$ with positive probability.

On the other hand, similar reasoning leads to

$$U (B, b) > U (B, a) \geq U (A, a)$$

and the two inequalities contradict each other. Hence it cannot be that neither type votes sincerely. □

**Lemma 9** There cannot be an equilibrium in which both types always vote for the same candidate.

**Proof.** Suppose that all voters vote for $A$ (say). Then we have that

$$U (A, a) > U (A, b) \geq U (B, b) > U (B, a)$$

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Moreover, since \( b \) types participate,

\[
U(A, b) = q(\alpha | b) \Pr[Piv_A | \alpha] - q(\beta | b) \Pr[Piv_A | \beta] \\
= q(\alpha | b) \frac{1}{2} e^{-n(rp_a + (1 - r)p_b)} - q(\beta | b) \frac{1}{2} e^{-n((1 - s)p_a + sp_b)} \\
\geq 0
\]

since the only circumstances in which a vote for \( A \) is pivotal is if no one else shows up. Since \( r > 1 - s \), a necessary condition for this to hold is that

\[
rp_a + (1 - r)p_b < (1 - s)p_a + sp_b
\]

We claim that

\[
U(B, b) - U(A, b) > 0
\]

which is equivalent to

\[
q(\beta | b) (\Pr[Piv_A | \beta] + \Pr[Piv_B | \beta]) > q(\alpha | b) (\Pr[Piv_B | \alpha] + \Pr[Piv_B | \alpha])
\]

Notice that

\[
\Pr[Piv_A | \beta] + \Pr[Piv_B | \beta] = e^{-n((1 - s)p_a + sp_b)} (1 + \frac{1}{2} n ((1 - s)p_a + sp_b)) \\
\Pr[Piv_B | \alpha] + \Pr[Piv_B | \alpha] = e^{-n(rp_a + (1 - r)p_b)} (1 + \frac{1}{2} n (rp_a + (1 - r)p_b))
\]

and the first term is greater since the function

\[
g(x) = e^{-x} (1 + \frac{1}{2} x)
\]

is decreasing for \( x > 0 \) and \( rp_a + (1 - r)p_b < (1 - s)p_a + sp_b \).

Thus,

\[
U(B, b) - U(A, b) > 0
\]

which contradicts the assumption that \( b \) types vote for \( A \). \( \blacksquare \)

Lemmas 8 and 9 imply that any equilibrium must have the following form: one type votes sincerely and the other type votes sincerely with positive probability.

If \( a \) types vote sincerely and \( b \) types vote sincerely with probability \( \mu > 0 \), then

\[
\begin{align*}
\sigma_A &= rp_a + (1 - r)(1 - \mu)p_b; \\
\sigma_B &= (1 - r)\mu p_b \\
\tau_A &= (1 - s)p_a + s(1 - \mu)p_b; \\
\tau_B &= sp_b
\end{align*}
\]

(23)

If \( b \) types vote sincerely and \( a \) types vote sincerely with probability \( \mu > 0 \), then

\[
\begin{align*}
\sigma_A &= \tau \mu p_a; \\
\sigma_B &= r(1 - \mu)p_a + (1 - r)p_b \\
\tau_A &= (1 - s)\mu p_a; \\
\tau_B &= (1 - s)(1 - \mu)p_a + sp_b
\end{align*}
\]

(24)

We will now show that when \( n \) is large, both types vote sincerely.

As a first step we have,

**Lemma 10** In any sequence of equilibria, the threshold costs tend to zero; that is, \( \limsup c_a(n) = \limsup c_b(n) = 0 \).
\textbf{Proof.} Suppose to the contrary, that for some sequence, $\lim c_a (n) > 0$. In that case, the gross benefits (excluding the costs of voting) to voters with $a$ signals from voting must be positive; that is
\[
\lim (q (\alpha | a) \Pr[Piv_A | \alpha] - q (\beta | a) \Pr[Piv_A | \beta]) > 0
\]
where it is understood that the probabilities depend on $n$.

We know that along the given sequence, $\lim p_a (n) > 0$. By continuity, Lemma 9 implies that there is a lower bound, say $\mu > 0$, such that an $a$ type votes for $A$ with at least probability $\mu$. This implies that $\lim \sigma_A (n) \geq \lim r \mu p_a (n) > 0$ also and hence $\lim (n \sigma_A) = \infty$.

The remainder of the proof is the same as that in Lemma 2. \hfill $\blacksquare$

\textbf{Lemma 11} Suppose that there is a sequence of equilibria such that $\lim (n \sqrt{\sigma_A \sigma_B}) = \infty$. Then for large $n$,
\[
\frac{\Pr[Piv_B | \alpha]}{\Pr[Piv_B | \beta]} > \frac{\Pr[Piv_A | \alpha]}{\Pr[Piv_A | \beta]}
\]

\textbf{Proof.} Since $\lim (n \sqrt{\sigma_A \sigma_B}) = \infty$, we also have $\lim (n \sqrt{\sigma_A \sigma_B}) = \infty$. Then the Myerson approximations imply that, for large $n$, the required inequality is equivalent to
\[
\frac{1 + \sqrt{\tau_B \tau_A}}{1 + \sqrt{\tau_A \tau_B}} > \frac{1 + \sqrt{\sigma_B \sigma_A}}{1 + \sqrt{\sigma_A \sigma_B}}
\]
and a sufficient condition for this to hold is that
\[
\frac{\tau_B}{\tau_A} > \frac{\sigma_B}{\sigma_A}
\]

Suppose that for large $n$, we have an equilibrium in which $a$ types vote sincerely while $b$ types vote for $B$ with probability $\mu < 1$. Then the last inequality is equivalent to
\[
\frac{s \mu p_b}{(1 - s) p_a + s (1 - \mu) p_b} > \frac{(1 - r) \mu p_b}{r p_a + (1 - r) (1 - \mu) p_b}
\]
and this holds since $r$ and $s$ are both greater than $\frac{1}{2}$.

On the other hand, suppose that $b$ types vote sincerely while $a$ types vote for $A$ with probability $\mu < 1$. Then the inequality is equivalent to
\[
\frac{(1 - s) p_a (1 - \mu) + s p_b}{(1 - s) p_a \mu} > \frac{r p_a (1 - \mu) + (1 - r) p_b}{r p_a \mu}
\]
and this also holds since $r$ and $s$ are greater than $\frac{1}{2}$. \hfill $\blacksquare$

\textbf{Lemma 12} Suppose that there is a sequence of equilibria such that $\lim (n \sigma_A) < \infty$ and $\lim (n \sigma_B) < \infty$. Then for large $n$,
\[
\frac{\Pr[Piv_B | \alpha]}{\Pr[Piv_B | \beta]} > \frac{\Pr[Piv_A | \alpha]}{\Pr[Piv_A | \beta]}
\]
**Proof.** Since \( \lim (n\sigma_A) < \infty \) and \( \lim (n\sigma_B) < \infty \), we also have \( \lim (n\tau_A) < \infty \) and \( \lim (n\tau_B) < \infty \). This implies that all pivotal probabilities are positive, even in the limit.

From Lemma 10, we have that

\[
q(\alpha | a) \lim \Pr [Piv_A | \alpha] - q(\beta | a) \lim \Pr [Piv_A | \beta] = 0
\]
\[
q(\beta | b) \lim \Pr [Piv_B | \beta] - q(\alpha | b) \lim \Pr [Piv_B | \alpha] = 0
\]

and so

\[
\lim \frac{\Pr [Piv_B | \alpha]}{\Pr [Piv_B | \beta]} = \frac{q(\beta | b)}{q(\alpha | b)} > 1 > \frac{q(\beta | a)}{q(\alpha | a)} = \lim \frac{\Pr [Piv_A | \alpha]}{\Pr [Piv_A | \beta]}
\]

\[\blacksquare\]

**Lemma 13** In any sequence of equilibria, when \( n \) is large, voting is sincere.

**Proof.** There are three cases to consider.

**Case 1:** \( \lim (n\sqrt{\sigma_A \sigma_B}) = \infty \).

From Lemma 8, we know that at most one type of voter can mix. Suppose that all \( a \) types vote for \( A \) while \( b \) types vote for \( B \) with probability \( \mu < 1 \).

Let \( p_a > 0 \) and \( p_b > 0 \) be the equilibrium participation rates.

Again, define \( U(B,b) \) to be the gross payoff of voting for \( B \) for a voter with a \( b \) signal and similarly, define \( U(A,b) \). Since \( b \) type voters mix, it must be that they are indifferent. Moreover, since some \( b \) types participate, it must be that this payoff is positive. Thus we have,

\[ U(A,b) = U(B,b) > 0 \]

Thus the gross payoff to a \( b \) type from voting is

\[ U(B,b) = q(\beta | b) \Pr [Piv_B | \beta] - q(\alpha | b) \Pr [Piv_B | \alpha] > 0 \]

where the pivot probabilities are computed using the voting probabilities \( \sigma_A = rp_a + (1 - r)(1 - q)p_b, \sigma_B = (1 - r)qp_b, \tau_A = (1 - s)p_a + s(1 - q)p_b \) and \( \tau_B = sqp_b \).

The inequality \( U(B,b) > 0 \) may be rewritten as

\[
\frac{\Pr [Piv_B | \beta]}{\Pr [Piv_B | \alpha]} > \frac{q(\alpha | b)}{q(\beta | b)}
\]

Lemmas 11 then implies that for large \( n \),

\[
\frac{\Pr [Piv_A | \beta]}{\Pr [Piv_A | \alpha]} > \frac{q(\alpha | b)}{q(\beta | b)}
\]

which is equivalent to

\[ U(A,b) = q(\alpha | b) \Pr [Piv_A | \alpha] - q(\beta | b) \Pr [Piv_A | \beta] < 0 \]

which is a contradiction.
A similar argument applies if it is the \( a \) types that mix.

**Case 2:** \( \lim (n\sigma_A) < \infty \) and \( \lim (n\sigma_B) < \infty \).

The argument is the same as in case 1 except that it now Lemma 12 that leads to the ratio inequality.

**Case 3:** (i) \( \lim (n\sigma_A) = \infty \) or \( \lim (n\sigma_B) = \infty \).

Suppose that in fact, \( \lim (n\sigma_A) = \infty \) while \( \lim (n\sigma_B) < \infty \). Then from (23) and (24) it follows that, for large \( n \), \( a \) types vote sincerely and \( b \) types mix. Moreover, we also have \( \lim (n\sqrt{\sigma_A\tau}) < \infty \), \( \lim (n\sigma_A) = \infty \) or \( \lim (n\tau_B) < \infty \).

Observe that

\[
\frac{\Pr[Piv_A | \alpha] + \Pr[Piv_B | \alpha]}{\Pr[Piv_A | \beta] + \Pr[Piv_B | \beta]} = \frac{e^{-n(\sigma_A + \sigma_B)} \left( I_0 \left( 2n\sqrt{\sigma_A\sigma_B} \right) + \frac{1}{2} \left( \sqrt{\frac{n}{\sigma_A}} + \sqrt{\frac{n}{\sigma_B}} \right) I_1 \left( 2n\sqrt{\sigma_A\sigma_B} \right) \right)}{e^{-n(\tau_A + \tau_B)} \left( I_0 \left( 2n\sqrt{\tau_A\tau_B} \right) + \frac{1}{2} \left( \sqrt{\frac{n}{\tau_A}} + \sqrt{\frac{n}{\tau_B}} \right) I_1 \left( 2n\sqrt{\tau_A\tau_B} \right) \right)}
\]

and since \( \lim (n\sqrt{\tau_A\tau_B}) < \infty \), the asymptotic behavior of this term is determined solely by the value of \(-n(\sigma_A + \sigma_B - \tau_A - \tau_B)\). If this is positive, the ratio is unbounded. If this is negative, then the ratio goes to zero. In either case, for large \( n \), a voter cannot be indifferent between voting for \( A \) and voting for \( B \) and so would not mix between the two.

We have thus shown that all equilibria must involve sincere voting.

It remains to show that given sincere voting, there is a unique set of participation rates—that is, there is a unique solution \((p_a^*, p_b^*)\) to \( IR_a \) and \( IR_b \). As we show next, this is also true in large elections.

**Lemma 14** In large elections, there is a unique solution to the cost threshold equations \( IR_a \) and \( IR_b \).

**Proof.** In this appendix we establish Lemma 14; that is, when \( n \) is large, there is a unique solution to the equations

\[
U_a (p_a, p_b) \equiv q (\alpha | a) \Pr[Piv_A | \alpha] - q (\beta | a) \Pr[Piv_A | \beta] = F^{-1} (p_a) \quad (\text{IRa})
\]

\[
U_b (p_a, p_b) \equiv q (\beta | b) \Pr[Piv_B | \beta] - q (\alpha | b) \Pr[Piv_B | \alpha] = F^{-1} (p_b) \quad (\text{IRb})
\]

that determine equilibrium cost thresholds.

Specifically, we will show that, at any intersection of the two, the curve determined by \( IR_a \) is steeper than that determined by \( IR_b \), that is,

\[
- \left( \frac{\partial U_a}{\partial p_a} - (F^{-1} (p_a))' \right) \div \frac{\partial U_a}{\partial p_b} > - \left( \frac{\partial U_b}{\partial p_a} - (F^{-1} (p_b))' \right) \div \frac{\partial U_b}{\partial p_b}
\]

This result does not hold in a corresponding model of costly voting with a fixed population. Ghosal and Lockwood (2007) provide an example with the majority rule in which there are multiple cost thresholds and hence, multiple equilibria.
The calculation of the partial derivatives is facilitated by using the following simple fact. If we write,

$$Pr[(l, k) | \alpha] = e^{-nrPa} \frac{(nrPa)^l}{l!} e^{-n(1-r)Pb} \frac{(n(1-r)Pb)^k}{k!}$$

as the probability of outcome $(l, k)$ in state $\alpha$, then

$$\frac{\partial Pr[(l, k) | \alpha]}{\partial Pa} = nr Pr[(l-1, k) | \alpha] - nr Pr[(l, k) | \alpha]$$

$$\frac{\partial Pr[(l, k) | \alpha]}{\partial Pb} = n(1-r) Pr[(l, k-1) | \alpha] - n(1-r) Pr[(l, k) | \alpha]$$

Similar expressions obtain for the partial derivatives of $Pr[(l, k) | \beta]$.

Since the probability of a pivotal term $Piv_C$ where $C = A, B$ is just a linear combination of terms of the form $Pr[(l, k) | \alpha]$, we obtain

$$\frac{\partial Pr[Piv_C | \alpha]}{\partial Pa} = nr Pr[Piv_C - (1, 0) | \alpha] - nr Pr[Piv_C | \alpha]$$

$$\frac{\partial Pr[Piv_C | \alpha]}{\partial Pb} = n(1-r) Pr[Piv_C - (0, 1) | \alpha] - n(1-r) Pr[Piv_C | \alpha]$$

Again, similar expressions obtain for the partial derivatives of $Pr[Piv_C | \beta]$ where $C = A, B$.

Myerson (2000) has shown that when the expected number of voters is large, the probabilities of the “offset” events in state $\alpha$ are

$$Pr[Piv_C - (1, 0) | \alpha] \approx Pr[Piv_C | \alpha] x^{\frac{1}{2}}$$

$$Pr[Piv_C - (0, 1) | \alpha] \approx Pr[Piv_C | \alpha] x^{-\frac{1}{2}}$$

where

$$x = \frac{1-r \times Pb}{r \times Pa}$$

Similarly, the probabilities of the offset events in state $\beta$ are

$$Pr[Piv_C - (1, 0) | \alpha] \approx Pr[Piv_C | \beta] y^{\frac{1}{2}}$$

$$Pr[Piv_C - (0, 1) | \alpha] \approx Pr[Piv_C | \beta] y^{-\frac{1}{2}}$$

where

$$y = \frac{s \times Pb}{1-s \times Pa}$$

Using Myerson’s offset formulae it follows that

$$\frac{\partial U_a}{\partial Pa} \approx nq(\alpha | a) r Pr[Piv_A | \alpha] (x^{\frac{1}{2}} - 1) - nq(\beta | a) (1-s) Pr[Piv_A | \beta] (y^{\frac{1}{2}} - 1)$$

$$\frac{\partial U_a}{\partial Pb} \approx nq(\alpha | a) (1-r) Pr[Piv_A | \alpha] (x^{-\frac{1}{2}} - 1) - nq(\beta | a) s Pr[Piv_A | \beta] (y^{-\frac{1}{2}} - 1)$$
and similarly,
\[
\frac{\partial U_b}{\partial p_a} \approx nq(\beta \mid b)(1 - s) \Pr[Piv_B \mid \beta](y_b^{\frac{1}{2}} - 1) - nq(\alpha \mid b)r\Pr[Piv_B \mid \alpha](x_b^{\frac{1}{2}} - 1)
\]
\[
\frac{\partial U_b}{\partial p_b} \approx nq(\beta \mid b)s\Pr[Piv_B \mid \beta](y_b^{\frac{1}{2}} - 1) - nq(\alpha \mid b)(1 - r)\Pr[Piv_B \mid \alpha](x_b^{\frac{1}{2}} - 1)
\]

We have argued that when \( n \) is large, any point of intersection of IR\( a \) and IR\( b \), say \((p_a, p_b)\), results in efficient electoral outcomes—\( A \) wins in state \( \alpha \) and \( B \) wins in state \( \beta \). This requires that \((p_a, p_b)\) lie in the efficiency cone determined by
\[
\frac{1 - r p_b}{r} < 1 \text{ and } \frac{s p_b}{1 - s p_a} > 1
\]
and by definition this is the same as
\[
x < 1 \text{ and } y > 1
\]

From this it follows that at any point \((p_a, p_b)\) satisfying (26),
\[
\frac{\partial U_a}{\partial p_a} < 0 \text{ and } \frac{\partial U_a}{\partial p_b} > 0
\]
and similarly,
\[
\frac{\partial U_b}{\partial p_a} > 0 \text{ and } \frac{\partial U_b}{\partial p_b} < 0
\]

Thus at any \((p_a, p_b)\) satisfying (26), the curves determined by IR\( a \) and IR\( b \) are both positively sloped.

Since \((F^{-1}(p_a))'\) and \((F^{-1}(p_b))'\) are both positive, in order to establish the inequality in (25), it is sufficient to show that
\[
\left( -\frac{\partial U_a}{\partial p_a} \right) \div \left( -\frac{\partial U_a}{\partial p_b} \right) > \left( -\frac{\partial U_b}{\partial p_a} \right) \div \left( -\frac{\partial U_b}{\partial p_b} \right)
\]
which is equivalent to
\[
\frac{q(\alpha \mid a)r\Pr[Piv_A \mid \alpha](1 - x_b^{\frac{1}{2}}) + q(\beta \mid a)(1 - s)\Pr[Piv_A \mid \beta](y_b^{\frac{1}{2}} - 1)}{q(\alpha \mid a)(1 - r)\Pr[Piv_A \mid \alpha](x_b^{\frac{1}{2}} - 1) + q(\beta \mid a)s\Pr[Piv_A \mid \beta](1 - y_b^{\frac{1}{2}})} > \frac{q(\alpha \mid b)r\Pr[Piv_B \mid \alpha](1 - x_b^{\frac{1}{2}}) + q(\beta \mid b)(1 - s)\Pr[Piv_B \mid \beta](y_b^{\frac{1}{2}} - 1)}{q(\alpha \mid b)(1 - r)\Pr[Piv_B \mid \alpha](x_b^{\frac{1}{2}} - 1) + q(\beta \mid b)s\Pr[Piv_B \mid \beta](1 - y_b^{\frac{1}{2}})}
\]

Using
\[
q(\alpha \mid a) = \frac{r}{r + (1 - s)} \text{ and } q(\beta \mid b) = \frac{s}{s + (1 - r)}
\]
and writing
\[
L_A = \frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} \text{ and } L_B = \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]}
\]

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as the two likelihood ratios, the inequality above is the same as
\[
\frac{(r)^2 (1 - x^{\frac{1}{2}}) + (1 - s)^2 (y^{\frac{1}{2}} - 1) L_A}{r (1 - r) (x - y^{\frac{1}{2}} - 1) + s (1 - s) (y^{\frac{1}{2}} - 1) L_A} > \frac{r (1 - r) (1 - x^{\frac{1}{2}}) + s (1 - s) (y^{\frac{1}{2}} - 1) L_B}{(1 - r)^2 (x - y^{\frac{1}{2}} - 1) + s (1 - y^{\frac{1}{2}}) L_B}
\]

Cross-multiplying and cancelling terms, further reduces the inequality to
\[
\left(\frac{(1 - r)(1 - s)}{rs}(x^{\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{\frac{1}{2}})\right) \times L_A
> \left(\frac{(1 - r)(1 - s)}{rs}(x^{\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{\frac{1}{2}})\right) \times \frac{rs}{(1 - r)(1 - s)} \times L_B
\]

(28)

We claim that for all \((p_a, p_b)\) in the efficiency cone,
\[
\frac{(1 - r)(1 - s)}{rs}(x^{\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{\frac{1}{2}}) < 0
\]

(29)

To see this, note that by definition,
\[
y = \frac{s p_b}{1 - s p_a} = \frac{r s}{(1 - r)(1 - s)} \frac{1 - r p_b}{r p_a}
= \frac{r s}{(1 - r)(1 - s)} x
= R x
\]

where \(R = \frac{r s}{(1 - r)(1 - s)}\). Substituting \(y = R x\) we obtain
\[
R(x^{\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{\frac{1}{2}}) = R^{-1}(x^{\frac{1}{2}} - 1)(R^\frac{1}{2} x^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - R^{-\frac{1}{2}} x^{\frac{1}{2}})
\]

Now consider the function
\[
\phi(x) = R^{-1}(x^{\frac{1}{2}} - 1)(R^\frac{1}{2} x^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - R^{-\frac{1}{2}} x^{\frac{1}{2}})
\]
Since \(x < 1 < y = R x\), we have \(R^{-1} < x < 1\). Notice that \(\phi(1) = 0 = \phi(R^{-1})\). It is routine to verify that \(\phi\) is convex and so \(\phi(x) < 0\) for all \(x \in (R^{-1}, 1)\). Thus we have established (29).

Now because of (29), the inequality in (28) reduces to
\[
\frac{\Pr[Piv_A | \beta]}{\Pr[Piv_A | \alpha]} < R \times \frac{\Pr[Piv_B | \beta]}{\Pr[Piv_B | \alpha]}
\]

(30)

Finally, notice that IRa and IRb imply, respectively, that
\[
\frac{r}{1 - s} = \frac{q(\alpha | a)}{q(\beta | a)} > \frac{\Pr[Piv_A | \beta]}{\Pr[Piv_A | \alpha]} \quad \text{and} \quad \frac{q(\alpha | b)}{q(\beta | b)} > \frac{\Pr[Piv_B | \beta]}{\Pr[Piv_B | \alpha]}
\]
and this immediately implies (30), thereby completing the proof that at any point of intersection of IRa and IRb, the slope of IRa is greater than the slope of IRb. This means that the curves cannot intersect more than once and completes the proof of Lemma 14. ■
References


