Selling Information*

Johannes Hörner† and Andrzej Skrzypacz ‡

November 27, 2009

Abstract

We study a buyer-seller problem in which the good is information and there are no property rights. The buyer is reluctant to pay for information whose value he does not know, but the seller cannot credibly reveal this value without disclosing the information itself. Information comes as divisible hard evidence. We show how the seller can appropriate a substantial fraction of the value through gradual revelation, and how the entire value can be extracted with the help of an intermediary.

1 Introduction

This paper examines the following problem. A Firm has to make a choice between investing or not in a project. The optimal action depends on a state of the world that the Firm does not know. There is one Agent who knows the state of the world. Furthermore, if the state is such that investment is optimal, he can prove it, that is, he can produce evidence in favor of the state (we call such Agent the “Invest” type or type-1). Producing a piece of evidence in favor of investment might not rule out the other state of the world entirely, so that the Agent can

---

*We thank Daron Acemoglu, Drew Fudenberg, David Kreps, R.Vijay Krishna, Romans Pancs and seminar participants at Harvard-MIT, Stanford, Yale, SED 2009, Einaudi Institute in Rome, Turin and Florence for useful comments and suggestions.

†Yale University, 30 Hillhouse Ave., New Haven, CT 06520, USA. johannes.horner@yale.edu.

‡Stanford University, Graduate School of Business, andy@gsb.stanford.edu.
also provide partial evidence. That is, those arguments can be sometimes also provided by an
Agent that knows that not investing is optimal (we call him the “Not Invest” type or type-0).
The Agent does not care about which action the Firm eventually takes, but he cares about any
monetary payment he could possibly extract for the information he owns.

We ask if the Agent can sell the information to the Firm and in particular, what is the most
an Agent can get if he knows that the Firm should invest, when the Firm’s prior belief is such
that the Firm would prefer not to.

The problem is non-trivial because we assume that the two parties cannot write a binding
contract with payments that are contingent on the information released by the Agent. Once the
Firm knows the information, the Agent cannot take that information away, and the Firm can
use it as it wishes. Therefore, once the Agent conclusively proves that investment is optimal, the
Firm will no longer agree to any payment to the Agent. On the other hand, the Firm is reluctant
to pay for information whose value is still uncertain. But the only way for the Agent to credibly
reveal the value of its information to the Firm is to disclose it. This is known as the Arrow’s
Information Paradox (Arrow, 1959): information, unlike physical goods, cannot be taken away.

We model the interaction between the Agent and the Firm prior to the Firm’s decision as
a game with possibly arbitrarily many rounds of information disclosure and payments made
between the players. We treat information as perfectly divisible and in the first part of the paper
we allow the Agent to send a signal that is a “partial proof” or a “piece of a puzzle,” formally
modeled as a signal technology that can either increases the firm’s belief that investment is
profitable or reduce to zero (with the reduction being possible only if investment is in fact not
profitable). In the second part of the paper we allow for a trusted intermediary/signaling device
that can send credible signals arbitrarily correlated with the Agent’s knowledge.

Our main results under the first signal technology can be summarized as follows:

1) **Information is valuable:** There exist equilibria of our game in which the Firm pays the
Agent (and the Agent discloses the information in return). There exist equilibria with positive
payments, even when there is only one round of communication, and without resorting to partial
proofs. In that case however, the “Invest” and “Not Invest” types earn the same equilibrium
payoff.
2) The “Invest” type gains from using partial proofs, and releasing information gradually: If the game has a finite number of rounds of communication, the highest equilibrium payoff to the “Invest” type strictly increases in the number of rounds. To put it differently, the “Invest” Agent is always better off when he sells information that changes the prior belief from \( p \) to \( p' > p \) in two steps rather than in one.

3) The best equilibrium (for the “Invest type”) involves an initial burst of information revelation given away for free, followed by gradual information sales: We explicitly solve for the equilibrium that maximizes the “Invest” type’s payoff for any number of rounds of communication. As the number of rounds gets large, in the best equilibrium, the Agent first sends for free a signal that makes the Firm indifferent between investing or not, and then slowly sells the remaining information. In order to maximize his payoff, the “Invest” Agent wants to separate from the “Not Invest” type, but he cannot separate too fast, since then the Firm would stop paying him.

4) The best “Invest” Agent payoff is independent of the Firm’s prior belief (if the prior is such that the Firm should not invest) and the payoff is less than the full value of information: Since the optimal strategy calls for revealing the first big chunk of information for free, and the posterior that the first step induces is independent of the prior belief, the prior becomes irrelevant by the time the Firm starts paying the Agent (recall that we assume that the Firm’s prior belief is such that not investing is optimal). We also show that if the prior of the Firm is not to invest, then the payoff of the “Invest” type is necessarily strictly less than the value of his information.

In the second part of the paper we ask whether there is any value in having a mediator, or intermediary, whose authority is rather limited: in every round, the intermediary allows the Agent to commit to any randomization over information disclosures (even if the Agent is not indifferent over the different signals that the intermediary might randomize over). In this way, the intermediary can generate any posterior belief in the unit interval, and the posterior belief after a given signal might fall below the prior belief without necessarily dropping to zero, unlike in our baseline model. The intermediary has no other authority, and can be therefore be thought of as a machine or a computer program. Its only role is that it allows us to treat mixed strategies
in the baseline model as observable pure strategies.

5) **With an intermediary, the Agent of the “Invest” type can extract the entire value of the information:** We prove that, by proceeding in arbitrarily many steps, there is an equilibrium strategy in which the Agent of the “Invest type” can extract an amount arbitrarily close to the full value of the information. We prove this without actually describing the asymptotically optimal strategy, which cannot be solved for in closed-form.

This paper is related to several strands of literature. The gradualism that appears in equilibrium is related to findings of the literature on contribution games. See, for instance, Admati and Perry (1987), Marx and Matthews (2000) and Compte and Jehiel (2004). See also Gul (2001) and Che and Sákovics (2004) for the dynamic resolution of the hold-up problem. Indeed, there are similarities between contributing to a public good, and giving information. In both cases, concessions are irreversible. However, unlike in the public goods case, there is a strong asymmetry between the players here, and the payoff structure is quite different. As a result, some of the findings are different as well. For instance, there is no counterpart in that literature to what happens in the first round here, in which a big chunk of information is given away for free. Gradualism appears here only afterwards.

The constraints on information revelation are reminiscent of the literature on long cheap talk. See, in particular, Forges (1990) and Aumann and Hart (2003). As is the case here, the problem is how to “split” a martingale optimally over time. That is, the Firm’s belief is a martingale, and the optimal strategy specifies its distribution over time. The similarity is especially clear in the case of an intermediary. There are important differences, however. In particular, unlike in that literature, information revelation does not only occur prior to the choice of payoff-relevant actions, since the Firm pays the Agent as information gets revealed over time. In fact, payments are allowed to go in either direction in our model, as we do not constrain payments from the Firm to the Agent to be positive. As in Forges and Koessler (2008), messages here are type-dependent, as the “No Invest” Agent is constrained in the messages he can send.

Finally, there is a vast literature directly related to the value of information. See, among others, Admati and Pfleiderer (1988 and 1990). Eső and Szentes (2007) and Gentzkow and Kamenica (2009) take a mechanism design approach to this problem.
2 Splitting Information: A Simple Example

Suppose that an Agent may own a valuable piece of information for another party, the Firm. More precisely, she may either know nothing, which we denote by \( \iota = \emptyset \), or one piece of information (say, \( \iota = A \)), or two pieces of information (say, \( \iota = A \cup B \)). The nature of this information is irrelevant. What matters here is that (i) the possible types of the Agent, i.e., the information she may have, admit a natural ordering, from a least informed type to a most informed type, (ii) the Firm’s payoff increases in the information she obtains. That is, denoting by \( v(\emptyset) \), \( v(A) \) and \( v(A \cup B) \) the payoff the Firm can secure if it acquires the corresponding information, we assume that \( v(\emptyset) < v(A) < v(A \cup B) \), and normalize \( v(\emptyset) \) to zero, which is also the payoff that the Firm obtains if the Agent refuses to disclose any information. Finally, it is also assumed here that the Agent does not care about the information that she discloses \textit{per se}, but only about the payments she can obtain from the Firm in exchange for it. The probabilities with which the Agent is of a given type is common knowledge. Let \( p(A) \) and \( p(A \cup B) \) denote these probabilities (so that the Agent is uninformed with probability \( 1 - p(A) - p(A \cup B) \)).

Suppose first that the Agent has the opportunity to sell this information to the Firm in one shot. How much is the Firm willing to pay for the Agent’s potential information, assuming that the Agent will release all she knows if and only if the Firm makes this payment? Obtaining the information is worth in expectation

\[
p(A)v(A) + p(A \cup B)v(A \cup B)
\]

to the Firm, while not receiving any information is worth nothing to the Firm. Therefore, this value is also what the Firm is willing to pay for full disclosure.

Let us now assume instead that information is disclosed in two stages (without discounting), with payments occurring before each disclosure. More precisely, assume that the Agent discloses \( A \), if she can, in the first round, assuming that the Firm made the appropriate payment, in which case she also disclose \( B \) in the second period after another appropriate payment. How much is the Firm willing to pay at each stage? Let us solve the game backward: if \( A \) is disclosed, the Firm assigns probability \( p(A \cup B | A) \) to the event that the Agent also knows \( B \). Such knowledge
would be worth \( v(B) \), but knowing already \( A \), the Firm can secure \( v(A) \) anyhow. Therefore, she is willing to pay up to

\[
p(A \cup B | A)(v(B) - v(A))
\]

upfront for the possibility of additional information. The best for the Agent is then to ask for that much. This leaves however the Firm with a net continuation payoff of \( v(A) \) at the beginning of the second round. Therefore, in the first, it is willing to pay at most

\[
(p(A) + p(A \cup B))v(A)
\]

for the possibility of receiving this first piece of information. All in all, the Firm expects to pay

\[
(p(A) + p(A \cup B))(v(A) + p(A \cup B| A \text{ or } A \cup B)(v(B) - v(A))) = p(A)v(A) + p(A \cup B)v(A \cup B)
\]

over the course of the two rounds. Therefore, note that the Firm is indifferent over the two scenarios.

Consider, however, the fully informed Agent, who knows both \( A \) and \( B \). In the first scenario, she gets precisely the one payment of the Firm. In the second scenario, however, she knows that the second round will be reached with \( A \) being disclosed, and therefore she expects to obtain the sum of the Firm’s two payments:

\[
(p(A) + p(A \cup B))v(A) + p(A \cup B | A)(v(B) - v(A)),
\]

which is clearly larger than the payment she receives in one shot, since the second term is not multiplied by the probability \( p(A) + p(A \cup B) \). Therefore, an informed Agent gains from the information being sold slowly over time. Because the Firm is indifferent between both scenarios, and because in both scenarios all information gets disclosed, it must be that the uninformed Agent prefers the scenario in which information gets disclosed in one shot.

This example illustrates that splitting information might be a good idea from the point of view of more informed agent, and rewarding such agents might be desirable from an \textit{ex ante} perspective, if for instance, agents have to be given incentives to acquire information. But this
very simple model appears ill-suited to understand how information should be optimally disclosed when the information structure is richer. We turn to a richer but specific model next, before discussing how its ingredients affect the results (see subsection 4.4).

3 The Model

There are two risk-neutral players: an Agent and a Firm. There are two states of the world, \( \omega \in \Omega := \{0, 1\} \). The Agent is privately informed of the state of the world at the beginning of the game, but the Firm is not. The Firm’s prior belief that the state is 1 is \( p_0 \), which is common knowledge. The fact that the Agent is perfectly informed is a normalization. See below for how to adjust the analysis in case that she is not.

The game lasts \( K \) rounds, although our focus will be on what happens as \( K \) grows arbitrarily large. After the \( K \) rounds have elapsed, the Firm must take a binary action \( a = I, N \). Either the Firm chooses to “Invest” (\( I \)) or to “Not Invest” (\( N \)). Not investing yields a safe (i.e., state-independent) payoff normalized to 0. Investing yields a payoff 1 when the state of the world is \( \omega = 1 \) and \(-\gamma\) if the state of the world is \( \omega = 0 \) (for some \( \gamma > 0 \)). That is, the “Investing” action is risky: it can pay more than the safe action, but only in one state.\(^1\) The parameter \( \gamma \) measures the cost of taking this action, if it is inappropriate.\(^2\) Since the agent knows the state, we call him the “Invest” type, or type-1 agent, if \( \omega = 1 \), and the “Not Invest” type, or type-0 agent, otherwise. Note that, absent any information revelation, the Firm’s optimal action is to invest if and only if

\[
p \geq p^* := \frac{\gamma}{1 + \gamma},
\]

and obtain thereby a payoff of

\[
w(p) := (p - (1 - p)\gamma^+),
\]

where \( a^+ := \max\{0, a\} \). While our analysis will cover both the case in which the prior belief \( p_0 \)

---

\(^1\)Since we normalized to one the payoff of the investment in state 1, comparative statics with respect to \( \gamma \) have more than one interpretation.

\(^2\)Alternatively, the firm has to choose between action \( A \) or \( B \). It gets a payoff 1 if the action-state combinations are \( \{A, 0\} \) or \( \{B, 1\} \) and a payoff 0 otherwise. That payoff structure is equivalent to the “Invest” decisions with \( \gamma = 1 \).
is below or above $p^*$, we shall implicitly assume the more interesting case in which $p_0$ is smaller than $p^*$, unless stated otherwise. The payoff $w(p)$ will be referred to as the Firm’s outside option, and we shall generalize the analysis to more general outside options in Subsection 4.4.

In each of the $K$ rounds before the action is taken, the Firm and Agent can make a monetary transfer, and the Agent can reveal some information if he wishes to. More precisely, the strategy has two parts. In round $k = 1, \ldots, K$, as a function of the history of transfers and information disclosures up to that point, the Agent and the Firm can simultaneously make a non-negative transfer $t^A_k$ and $t^F_k$, respectively, to the other party.\footnote{The Reader might wonder what is gained by allowing the Agent to pay the Firm. After all, it is the Agent who owns the unique valuable good in this model, which is the information. Indeed, as we shall see, no payment will be made by the Agent in the equilibrium of the baseline model. But such payments will play a critical role once more general mechanisms (i.e. signaling technologies) are allowed.} Second, once these transfers are made and observed, the Agent may disclose some verifiable information. That is, this information is type-dependent (not cheap talk). Given any belief $p \in (0, 1)$ that the Firm might assign to state 1, following some arbitrary history, and for any $p' \geq p$, there exists some piece of information whose disclosure would lead the Firm to update its belief to $p'$. That is, for any $\{p, p'\}$, there exists a piece of information $\iota(p, p') \in \mathcal{I}$ that the type-1 Agent owns for sure, but that the type-0 Agent owns only with probability

$$q = \frac{1 - p'}{p'} \frac{p}{1 - p}.$$  

This implies that, if the Agent is expected to disclose this piece of information, it follows from Bayes’ rule that the posterior belief assigned to $\omega = 1$ is equal to

$$\frac{p}{p + (1 - p)q} = p'.$$  

If the Agent fails to disclose this piece of information, while she was expected to do if she could, the Firm updates its belief to zero. Note that, considering the case $p' = 1$, there is some piece of information that the type-0 Agent does not own, that is, a “proof” that the state is 1.\footnote{A modeling issue arises at $p = 0$, irrelevant for our purposes. What if the Firm, after some history of transfers and disclosures, assigns probability 0 to $\omega = 1$, but the Agent then discloses this proof? Fortunately, our purpose is to identify the best equilibrium, not to characterize the set of all equilibria, and this distinction need not concern us here: the equilibrium we shall describe remains an equilibrium if it is required that players cannot switch away from probability 1 beliefs, and remains the best equilibrium if this requirement (not imposed by perfect Bayesian equilibrium) is dropped.}
The specific nature of this information and of the set $\mathcal{I}$ need not concern us here (some interpretations and additional formal definitions are provided below). What matters here is that this information is perfectly divisible—that is, there exists such a piece of information for each $p' \geq p$—and uniquely defined in equilibrium—so that, if there are two pieces of information that the type-0 Agent owns with the same probability $q$, the equilibrium specifies which one the Agent is meant to reveal. (Otherwise, recognizing that the type-0 Agent had an incentive to reveal whichever of the two pieces of information she actually owned, if any, the Firm would not update its belief to $p'$ after all).

The Agent does not care about the Firm’s decision per se. All she seeks to do is to maximize the sum of the net transfers she receives during the $K$ rounds. The Firm seeks to maximize the payoff from its decision after the $K$ rounds, net of the payments it made. There is no discounting, nor any other type of frictions during the $K$ rounds.

More formally, a history of length $k$ is a sequence

$$h_k = \{(t_{k'}, t_{k'}, r_{k'})\}_{k'=0}^{k-1},$$

where $(t_{k'}, t_{k'}, r_{k'}) \in \mathbb{R}_+^2 \times \mathcal{I}$. The set of all such histories is denoted $H_k$ (set $H_0 := \emptyset$). Given some final history $h_K$, the Agent’s payoff is simply the sum of all net transfers over time:

$$V(h_K) = \sum_{k=0}^{K-1} (\tau^F_k - \tau^A_k).$$

Given state $\omega$, the Firm’s overall payoff results from its action, as well as from the sum of net transfers. If the firm chooses the safe action, it gets

$$W(\omega, h_K, N) = \sum_{k=0}^{K-1} (\tau^A_k - \tau^F_k).$$

If instead the Firm decides to invest, it receives

$$W(\omega, h_K, I) = \sum_{k=0}^{K-1} (\tau^A_k - \tau^F_k) + 1 \cdot 1_{\omega=1} - \gamma \cdot 1_{\omega=0},$$

where

$$1_{\omega=1} = \begin{cases} 1 & \text{if } \omega = 1 \\ 0 & \text{otherwise} \end{cases}, \quad 1_{\omega=0} = \begin{cases} 1 & \text{if } \omega = 0 \\ 0 & \text{otherwise} \end{cases}.$$
where $1_A$ is the indicator function of the event $A$.

Note that in all three of these payoff expressions there is no discounting, disclosing information has no cost. Also, note that in the first expression the Agent is indifferent over the Firm’s eventual action.

A prior belief $p_0$ and a strategy profile $\sigma := (\sigma^F, \sigma^A)$ define a distribution over $(\omega, h_K, a)$, and we let $V(\sigma), W(\sigma)$, or simply $(V, W)$ denote the expected payoff of the Agent and the Firm, respectively, relative to this distribution. When the strategy profile is understood, we also write $V(h_k), W(h_k)$ for the players’ continuation payoffs, given history $h_k$. We further write $V_0(\sigma)$ or $V_1(\sigma)$, or simply $V_0, V_1$, for the payoff of the Agent, when we condition on the state $\omega = 0, 1$. That is, $V_1(\sigma)$ is the payoff of type-1 Agent, when the strategy profile is $\sigma$.

Our focus is to identify the (perfect Bayesian) equilibrium $\sigma$ that maximizes the payoff of the type-1 agent. More precisely, we are interested in the limit of this equilibrium payoff as the number of rounds $K$ becomes arbitrarily large.\(^5\) We also discuss shortly some other equilibria (see Lemma 1).

One possible motivation for focusing on this equilibrium is that the probability of the state 1 is itself the result of some effort by the Agent. For instance, it could be that the action consists in hiring the Agent or not, and the two states correspond to the events in which the Agent is competent or not. Hiring the Agent is profitable if and only if the Agent is competent, and it might be socially desirable to reward activities, such as education, that increase the probability that the Agent is competent. With this interpretation, our model can be interpreted as a model of selling expertise. We embed our model in such a framework in Section 4. However, there is no need to do so at this stage.

As mentioned at the beginning of this section, that the Agent is perfectly informed of the state is irrelevant for our analysis, as long as, if she is of type 1, it is optimal for the Firm to invest, while it is optimal for him to not invest if she is of type 0. What matters then is the Firm’s belief about the Agent’s type. His outside option, as a function of this belief, has the same features as in our simpler model: for low enough beliefs, it is flat (at a level that can be normalized to 0), and it is increasing linearly for higher beliefs.

\(^5\)The equilibrium we shall obtain is also an equilibrium of the infinite-horizon, undiscounted game, but taking limits allows us to uniquely pin down the limiting strategy profile.
We conclude this subsection with a more formal description of strategies and equilibrium, which some readers might elect to skip.

A (behavior) strategy $\sigma^F$ for the Firm is a collection $(\{\tau^F_k\}_{k=0}^{K-1}, \alpha^F)$, where $\tau^F_k : H_k \rightarrow \mathbb{R}_+$, specifying a transfer $t^F_k := \tau^F_k(h_k)$ as a function of the history so far, as well as an action (a probability transition as well), $\alpha^F : H_K \rightarrow \{I, N\}$ after the $K$-th round.\(^6\)

A (behavior) strategy $\sigma^A$ for the Agent is a collection $(\{\tau^A_k, \iota^A_k\}_{k=0}^{K-1})$, where $\tau^A_k : H_k \rightarrow \mathbb{R}_+$ is a probability transition specifying the transfer $t^A_k := \tau^A_k(h_k)$ in round $k$ given the history so far and given the information she has, and $\iota^A_k : \mathcal{P}(I) \times H_k \rightarrow \mathcal{I}$ is a probability transition specifying the information that is released in round $k$, as a function of the state, the history up to the current round, and the transfers that were made in the round.\(^7\) As the Agent can only reveal what she knows, we impose $\iota^A_k(I_0, h_k, t^F_k, t^A_k) \in I_0$ for all $(h_k, t^F_k, t^A_k)$ and all $I_0$. Further, the possible sets of information are assumed to be sufficiently rich, so that, from every prior belief $p$, there exists a piece of information whose disclosure would lead to a posterior belief $p'$, for all $p' \in [p, 1]$.\(^8\)

The solution concept is perfect Bayesian equilibrium, as defined in Fudenberg and Tirole (1991, Definition 8.2).\(^9\)

---

\(^6\)Formally, $\mathcal{I}$ is some topological space, $\tau^F_k(h_k)$ is a probability distribution on $\mathbb{R}_+$, and for each Borel set $A \in \mathbb{R}_+$, $\tau^F_k(\cdot)[A]$ is a measurable function of $h_k$.

\(^7\)Her information consists of a subset $I_0$ of $\mathcal{I}$ (i.e., an element of $\mathcal{P}(\mathcal{I})$, the set of all subsets of $\mathcal{I}$) representing all the pieces of information she has. The type-1 agent has all the information, i.e. $I_0 = \mathcal{I}$.

\(^8\)Formally, the description of the game includes a Borel measure $\nu$ on $\mathcal{P}(\mathcal{I})$, representing the prior distribution over the sets of pieces of information owned by the type-0 Agent, with the property that, for every sequence $\{\iota_1, \ldots, \iota_k\}$ such that $\nu(I_0 : \{\iota_1, \ldots, \iota_k\} \in I_0) = (1-p)p_0/(p(1-p_0))$, then for every $p' \in [p, 1]$, there exists $\iota_{k+1} \in I$ such that $\nu(I_0 : \{\iota_1, \ldots, \iota_k, \iota_{k+1}\} \in I_0) = (1-p)p_0/(p'(1-p_0))$. We can pick, for instance, $I = [0,1]$, along with the Lebesgue measure on each unit interval. Note that this is formally a game of incomplete information with more than two types (in the sense of Harsanyi (1967-68)), as there are as many types as possible subsets of pieces of information the Agent might hold.

\(^9\)Fudenberg and Tirole define perfect Bayesian equilibria for finite games of incomplete information only. The suitable generalization of their definition to infinite games is straightforward and omitted.
4 Equilibrium Analysis

4.1 Preliminaries

For sake of simplicity, we restrict attention here to pure strategies, but all results are proved in the appendix without this restriction.

A pure strategy, then, calls for the Agent to disclose a specific piece of information at each round (revealing nothing being a special case). Of course, this is only possible if the Agent owns this piece of information. If she is of type 1, she does. But this is not necessarily the case if she is of type 0.

This implies that, from the Firm’s point of view, and ignoring the uninteresting case in which the Agent is supposed to reveal nothing, its posterior will take one of two values: either it will jump from \( p_0 \) up to some \( p' \), if the piece of information \( \iota(p_0, p') \) is indeed revealed. Or it will jump down to zero. This is illustrated in Figure 1 below. The two arrows indicate the two possible posterior beliefs. Note that, as a stochastic process, and viewed from the Firm’s perspective, this belief must follow a martingale: the Firm’s expectation of its posterior belief must be equal to its prior belief. This is not the case, however, from the Agent’s point of view. Given her knowledge of the state, she assigns different probabilities to these posterior beliefs than the Firm. If she is the type-1 Agent, she knows for sure that the belief will not decrease over time (so that the process is a submartingale relative to her information). Conditioning only on the state being 0, the expectation of the posterior belief is below \( p_0 \) (the process is then a supermartingale). But since the type-0 agent knows whether she has the information or not, she knows for sure whether the posterior will be \( p' \) or 0.

Note that the joint payoffs to the players cannot exceed the surplus in the game. If the probability of state 1 is \( p \), given the history \( h_k \), then the expected surplus (assuming optimal decision) is \( p \cdot 1 + (1 - p) \cdot 0 = p \). This means that the continuation payoffs must satisfy

\[
pV_1(h_k) + (1 - p)V_0(h_k) + W(h_k) \leq p. \tag{1}
\]

The equilibrium is efficient if this constraint is binding.

There are further constraints on equilibrium payoffs. From any history onward, the Agent
can secure a payoff of zero, independently of his type:

\[ V_1(h_k) \geq 0, V_0(h_k) \geq 0. \]

The Firm, on the hand, can secure a higher continuation payoff. If it receives no further information, it receives its outside option

\[ w(p) = (p - \gamma (1 - p))^+. \]  

(2)

Since additional information cannot hurt the Firm, this is lower a bound on \( W(h_k) \).

Our game admits a plethora of equilibria. For instance, there is an equilibrium in which no transfers are ever made, and no information is ever released. This equilibrium achieves the lower bounds on the players’ payoffs, and so provides a useful threat from any history onward, but it is clearly inefficient.

There exists also an efficient equilibrium in which no transfers are ever made, and type-1 reveals the state in the last period, so that the posterior belief is 1 with probability \( p \), and 0 otherwise. This yields a payoff of \( p \) to the Firm, and nothing to the Agent.

In fact, if there is an equilibrium with payoffs \((V_0, V_1)\) to the Agent, then there exists an efficient equilibrium with these payoffs, as the Agent can always disclose the state in the last period on the equilibrium path. This cannot weaken the incentives for the players to carry out the planned transfers, but it guarantees that the correct action is taken.

From the Agent’s point of view, we already know that her expected payoff \( V_0 + V_1 \) cannot
exceed $p - (p - \gamma (1 - p))^+$. Is there an equilibrium in which this payoff is achieved? One round is enough for this: the Firm pays this amount upfront, and the Agent reveals the state. If the Firm fails to pay, no information is disclosed. Note that the Firm is indifferent to pay, given the punishment for failing to do so, and the Agent is willing to release this information.

Given some equilibrium yielding payoffs $(V_0, V_1, W)$ in the game with $K$ rounds, we claim that

$$V_1 + W \leq \bar{V}_1 + (p - \gamma (1 - p))^+,$$

where $\bar{V}_1$ denotes the highest equilibrium type-1 Agent’s payoff. Otherwise, by simply starting from the equilibrium that yields $V_1$ to the type-1 Agent and $W$ to the Firm, and by increasing the initial transfer that the Firm is asked to make by $W - (p - \gamma (1 - p))^+$, we would obtain an equilibrium in which the type-1 Agent gets a payoff strictly above $\bar{V}_1$—a contradiction, by definition. Therefore, the equilibrium that maximizes the type-1 Agent’s payoff cannot leave any surplus to the Firm, and it also maximizes the sum of the Firm’s and type-1 Agent’s payoffs.

Last, because any efficient equilibrium must satisfy

$$pV_1(h_k) + (1 - p)V_0(h_k) + W(h_k) = p,$$

we claim that any equilibrium that maximizes $V_1$ also maximizes $V_1 - V_0$. To see this, note that we can assume without loss of generality that the former is efficient (by specifying full disclosure in the last round), so that

$$V_0 = \frac{p(1 - V_1) - W}{1 - p},$$

and so

$$V_1 - V_0 = \frac{V_1 + W - p}{1 - p}.$$

Therefore, maximizing the payoff difference $V_1 - V_0$ is equivalent to maximizing the sum $V_1 + W$, but as we have already remarked, this is in turn equivalent to maximizing $V_1$ only.

We summarize these observations in the following Lemma.

**Lemma 1** Given any history $h_k$, $k \leq K - 1$, the following holds for the continuation game:
1. There exists an equilibrium in which players are held to their minmax payoffs:

\[ W = (p - \gamma (1 - p))^+, V_1 = V_0 = 0. \]

2. There exists an equilibrium in which the Firm gets all the surplus:

\[ W = p, V_0 = V_1 = 0. \]

3. There exists an equilibrium in which the Agent receives all the surplus, net of the Firm’s minmax payoff:

\[ V = p - (p - \gamma (1 - p))^+, W = (p - \gamma (1 - p))^+. \]

4. If the agent receives \( V \) as a continuation equilibrium payoff, there exists an equilibrium in which the Firm receives all the residual surplus:

\[ W = p - V. \]

5. Any equilibrium that maximizes \( V_1 \), the type-1 Agent’s payoff, also maximizes \( V_1 + W \), the sum of the Firm’s and type-1 Agent’s payoffs, as well as \( V_1 - V_0 \), the difference between the two Agent’s types’ payoffs.

6. The set of equilibrium payoffs is non-decreasing in \( K \), the number of rounds.

The last conclusion is an immediate consequence from the fact that players can always choose not to make transfers, nor to disclose any information, in the first round. Note also that, since the type-1 Agent can always mimic the type-0 Agent, her payoff must be at least as high as the type-0’s payoff. This implies that the maximal equilibrium payoff for the type-0’s Agent is the one that maximizes the Agent’s ex ante payoff, as described above.

This leaves open our main concern: what is the highest equilibrium payoff, \( V_1 \), for the type-1 agent?
4.2 Benefits of Splitting Information

We now turn our focus to the equilibrium that maximizes the payoff of the “Invest” type of the Agent. We already know that it is possible for that Agent to appropriate some of the value of her information, but the question is whether she can get more than \( p \), which is just as much as the “Not Invest” type gets in the equilibrium we constructed so far. Unless the Agent can reveal the information slowly, we show that the answer is negative:

**Proposition 1** If \( K = 1 \), the highest equilibrium payoff to the “Invest” type is equal to \( p_0 - w(p_0) \). Moreover, in any equilibrium, both Agent’s types must receive the same payoff: \( V_0 = V_1 \).

**Proof.** With one round of communication, the payoff of the Agent can come only from the payment in the first (and only) round. Therefore, the payoffs of the two types of Agents have to be the same.

If, in equilibrium, the set of posterior beliefs of the Firm is \( \{0, p_1\} \), then the most the Firm is willing to pay is

\[
\mathbb{E}_F [w(p')] - w(p_0),
\]

where the subscript to the expectation refers to the fact that this is relative to the Firm’s belief, and \( p' \) is the the Firm’s posterior belief—a random variable with values in \( \{0, p_1\} \). Because beliefs must follow a martingale from the Firm’s point of view, it must be that the probability that the posterior is \( p_1 \) is \( \frac{p_1}{p} \), since

\[
p = \frac{p}{p_1} \cdot p_1 + \frac{p_1 - p}{p_1} \cdot 0.
\]

This means that the additional value from this information, relative to what the Firm can secure, is:

\[
\frac{p}{p_1} (p_1 - \gamma (1 - p_1))^+ - (p - \gamma (1 - p))^+.
\]

This is increasing in \( p_1 \) and so maximized at \( p_1 = 1 \), yielding a payment \( p - (p - \gamma (1 - p))^+ \). As we have already observed, this payoff can be supported as an equilibrium payoff. ■

The intuition is simple: with one round to go, the highest payment is achieved when the Firm’s next outside option—the payoff from the action it takes—is maximized, and this requires
full information disclosure on the equilibrium path. Note that, when $p < p^*$, the highest payoff in one round that the “Invest” Agent can get in equilibrium is simply the prior $p_0$.

It turns out that, with as little as two rounds, the “Invest” Agent can earn more. The key will be to reveal information gradually. Figure 2 represents one possible information disclosure strategy: in the first step, the Agent discloses the piece of information leading to a posterior belief of $p^*$ (or 0, if she fails to do so), and information is then fully disclosed in the second period. No payment is made in the first round. In the second round, the equilibrium of the one-round game is played. What is the resulting payoff to the type-1 agent? Note that the payoff of the one-round game, obtained in the second period, is now

$$p^* - (p^* - \gamma (1 - p^*))^+ = p^* > p_0.$$  

This argument is summarized in the following proposition.

**Proposition 2** Suppose that $p_0 < p^*$ and $K \geq 2$. Then the highest equilibrium payoff of the “Invest” type is at least $p^*$.

This illustrates the benefits of splitting information and revealing it slowly over time. Is the splitting that we described optimal with two periods to go? As it turns out, for $p_0 < p^*$, it is. But there are many other ways of splitting information with two periods to go that improve upon the one-round equilibrium, and among them, splits that also improve over the one-period equilibrium when $p_0 > p^*$. Allowing additional rounds will further improve what the type-1 Agent can achieve.
4.3 The highest payoff for the Agent of the “Invest” type

As we saw, the Agent’s “Invest” type may earn a higher payoff than the “Not Invest” type if it is possible to disclose information gradually. We now solve for the highest equilibrium payoff for the ”Invest” type, for every number of rounds $k$ and every prior belief $p_0$.

The best equilibrium involves a pure strategy by the Agent: the equilibrium is characterized by a sequence of pieces of information that she releases one by one, if she can. That is, the “Invest” type does disclose these pieces sequentially, and reveals the state in the last period, by disclosing all remaining pieces of information (that the “No Invest” cannot do). If the Agent is of the “No Invest” type, she discloses these pieces sequentially as well, until she can do so no longer. At this round, since she fails to disclose the requisite piece of information, the Firm learns that the Agent is uninformed, and updates the probability it assigns to state $\omega = 1$ to zero.

This equilibrium, then, can be summarized by a sequence of posterior beliefs, representing the Firm’s belief after each round that the state is 1, conditional on all pieces of information having been exhibited up that round. This sequence $\{p_0, \ldots, p_K\}$ starts at the Firm’s prior belief, $p_0$, and ends up at $p_K = 1$.

Of course, an equilibrium must also specify transfers, as well how players behave off the equilibrium path. The equilibrium is such that, from every round onward, and for every history up to this round, the Firm’s payoff is held down to its outside option. Therefore, if the firm’s belief in the next round is either $p_{k+1}$, or 0, given the current belief $p_k$, then the firm is willing to pay

$$\mathbb{E}_F[w(p')] - w(p_k),$$

where $p'$ is the random belief in the next round, with possible values 0 and $p_{k+1}$. The Agent does not make any transfers. In other words, the Agent extracts the maximal payment she can hope for from the Firm at every round. This sounds intuitive, but as we shall see, this will no longer be optimal when a slightly more general mechanism is considered.

If either player deviates from the specified course of actions, play reverts to the worst equilibrium, in which no further transfer is made, and no information is ever disclosed again. Since the equilibrium path strategies depend only on the number of remaining units and beliefs, we write
$V_{1,K}(p_0)$ to be the payoff of type-1 Agent given there are $K$ rounds remaining and the Firm’s belief is $p_0$.

The following proposition describes the optimal sequence of beliefs $\{p_0, \ldots, p_K\}$, and the payoff to the type-1 Agent, as a function of the number of rounds and the prior belief $p_0$.

**Proposition 3** The maximal equilibrium payoff of the type-1 Agent with $K$ rounds, and a prior belief $p_0$, is given by

$$V_{1,K}(p_0) = \begin{cases} \frac{K}{\gamma} (1 - \frac{1}{K} p_0^{1/K}) - (p_0 - \frac{\gamma}{1 - p_0})^+ & \text{if } p_0 \geq (p^*)^{\frac{K}{K-1}}, \\ V_{1,K-1}(p^*) & \text{if } p_0 < (p^*)^{\frac{K}{K-1}}. \end{cases}$$

On the equilibrium path, in the initial round, the type-1 Agent reveals a piece of information leading to a posterior belief of

$$p_1 = \begin{cases} p^\frac{k-1}{k} & \text{if } p \geq (p^*)^{\frac{K}{K-1}}, \\ p^* & \text{if } p < (p^*)^{\frac{K}{K-1}}, \end{cases}$$

after which the play proceeds as in the best equilibrium with $K - 1$ rounds, given prior $p_1$.

**Proof.** The proof is by induction on the number of rounds. The argument presented here is for pure strategies. See the appendix for the proof that there is no mixed-strategy equilibrium with a greater payoff to the type-1 Agent.

Our induction hypothesis is that, with $k \geq 1$ periods to go, and a prior belief $p = p_0$, the best equilibrium involves setting the next (non-zero) posterior, $p_1$, equal to $p_1 = p^{\frac{k-1}{k}}$ if $p^{\frac{k-1}{k}} \geq p^*$ (i.e. if $p \geq (p^*)^{\frac{k}{K-1}}$ for $k \geq 2$), and equal to $p^*$ otherwise.\(^{10}\) Further, the type-1 Agent’s maximal payoff with $k$ rounds to go is equal to

$$V_{1,k}(p) = \frac{k}{\gamma} (1 - p^{1/k}) - (p - \frac{\gamma}{1 - p})^+ \text{ if } p \geq (p^*)^{\frac{k}{K-1}}, \text{ and } V_{1,k}(p) = V_{1,k-1}(p^*) \text{ if } p < (p^*)^{\frac{k}{K-1}},$$

\(^{10}\)In this proof when we say that the equilibrium involves setting posterior $p_1$ we mean that from the type-1 Agent point of view the posterior will be $p_1$, while from the point of view of the Firm the posterior will be a random variable $p'$ which will take two values $\{0, p_1\}$.  

19
where \((x)^- := -\min\{0, x\} = x - x^+ \geq 0\). Note that this claim implies that

\[
V_{1,k}(p^*) = k \gamma \left(1 - (p^*)^{1/k}\right).
\]

Finally, as part of our induction hypothesis, we claim the following. Given some equilibrium, let \(X \geq 0\) denote the payoff of the Firm, net of its outside option, with \(k\) rounds left. That is, \(X := W_k(p) - w(p)\), where \(W_k(p)\) is the Firm’s payoff given the history leading to the equilibrium belief \(p\) with \(k\) rounds to go. Let \(V_{1,k}(p, X)\) be the maximal payoff of the type-1 Agent over all such equilibria, with associated belief \(p\), and excess payoff \(X\) promised to the Firm (let \(V_{1,k}(p, X) = -\infty\) if no such equilibrium exists). Then we claim that

\[
V_{1,k}(p, X) \leq V_{1,k}(p) + X.
\]

We first verify this with one round. As we have seen in Proposition 1, if \(K = 1\), the best is to set the posterior \(p_1\) equal to 1, which is indeed \(p^{K-1}\), the relevant specification given that \(p_1^{K-1} = 1 \geq p^*\). The payoff to the type-1 Agent is

\[
V_{1,1}(p) = p - (p - \gamma(1 - p))^+ = \gamma(1 - p) - (p - \gamma(1 - p))^-,
\]
as was to be shown. Note that this equilibrium is efficient. This implies that \(V_{1,1}(p, X) \leq V_{1,1}(p) + X\), for all \(X \geq 0\), as any additional payoff to the Firm must come as a reduction of the net transfer from the Firm to the Agent.

Assume that this holds with \(k\) rounds to go, and consider the problem with \(k + 1\) rounds. Of course, we do not know (yet) whether, in the continuation game, the Firm will be held down to its outside option.

Note that the Firm assigns probability \(p/p_1\) to the event that its posterior belief \(p'\) will be \(p_1\), since, by the martingale property, we have

\[
p = \mathbb{E}_F[p'] = \frac{p}{p_1} \cdot p_1 + \frac{p_1 - p}{p_1} \cdot 0.
\]

This implies that, with \(k + 1\) rounds, the Firm is willing to pay at most
\[ t^E_{k+1} := \frac{p}{p_1} \left( (p_1 - \gamma(1 - p_1))^+ + X' \right) - (p - \gamma(1 - p))^+, \]

where \( X' \) is the excess payoff of the Firm with \( k \) rounds to go, given posterior belief \( p_1 \). Therefore, the payoff to the type-1 Agent is at most

\[ V_{1,k+1}(p) \leq t^E_{k+1} + V_{1,k}(p_1; X') \leq \frac{p}{p_1} \left( (p_1 - \gamma(1 - p_1))^+ + X' \right) - w(p) + V_{1,k}(p_1) - X', \]

where the second inequality follows from our induction hypothesis. Note that, since \( p/p_1 < 1 \), this is a decreasing function of \( X' \): it is best to hold down the Firm to its outside option when the next round begins. Therefore, we maximize

\[ \frac{p}{p_1} (p_1 - \gamma(1 - p_1))^+ + V_{1,k}(p_1). \]

Note first that, given the induction hypothesis, all values \( p_1 \in [p, (p^*)^{\frac{k}{k-1}}) \) yield the same payoff, since for any such \( p_1 \), \( V_{1,k}(p_1) = V_{1,k-1}(p^*) \). The remaining analysis is now a simple matter of algebra. Note that, for \( p_1 \in [(p^*)^{\frac{k}{k-1}}, p^*) \) (which obviously requires \( p < p^* \)), the objective becomes (using the induction hypothesis)

\[ V_{1,k}(p_1) = k\gamma(1 - (p_1)^{1/k}) + (p_1 - \gamma(1 - p_1)), \]

which is increasing in \( p_1 \), so that the only candidate value for \( p_1 \) in this interval is \( p_1 = p^* \). Consider now picking \( p_1 \geq p^* \). Then we maximize

\[ \frac{p}{p_1} (p_1 - \gamma(1 - p_1))^+ + k\gamma(1 - p^{1/k}), \]

which admits a unique critical point \( p_1 = p^{\frac{k}{k+1}} \), achieving a payoff equal to \( (k+1)\gamma(1 - p^{1/(k+1)}) + p - \gamma(1 - p) = (k+1)\gamma(1 - p^{1/(k+1)}) \). Note, however, that this critical point satisfies \( p_1 \geq p^* \) if and only if \( p \geq (p^*)^{\frac{k+1}{k}} \).

Therefore, the candidates are \( \{p^*, \max\{p^*, p^{\frac{k}{k+1}}\}, 1\} \). Observe that setting the posterior belief \( p_1 \) equal to \( \max\{p^*, p^{\frac{k}{k+1}}\} \) does at least as well as choosing either \( p^* \) or 1. This establishes the
optimality of the strategy, and the optimal value for the type-1 Agent, with \( k + 1 \) rounds to go.

We finally must verify that \( V_{1,k+1}(p; X) \leq V_{1,k+1}(p) - X \). Given that we have observed that it is optimal to set \( X' = 0 \) in any case, any excess payoff to the Firm with \( k + 1 \) rounds to go is best obtained by a commensurate reduction in the net transfer from the Firm to the Agent in the first round (among the \( k + 1 \) rounds). This might violate individual rationality for some type of the Agent, but even if it does not, it still yields a payoff \( V_{1,k+1}(p; X) \) no larger than \( V_{1,k+1}(p) - X \) (if it does violate individual rationality, \( V_{1,k+1}(p; X) \) must be lower).

Note that, fixing \( p < p^* \), and letting \( k \to \infty \), it holds that \( p < (p^*)^{\frac{1}{k+1}} \) for all \( k \) large enough, so that, with enough rounds ahead, it is optimal to set \( p_1 = p^* \) in the first, and then to follow the sequence of posterior beliefs \((p^*)^{k+1}, (p^*)^{k+2}, \ldots, 1\). The payoff to the type-1 Agent from doing so tends to

\[
\lim_k V_{1,k}(p^*) = -\gamma \ln p^*,
\]

and the sequence of posteriors successively used becomes dense in \([p^*, 1]\). Therefore, with sufficiently many rounds, the equilibrium involves progressive disclosure of information, with a first big step leading to the posterior belief \( p^* \), given the prior belief \( p_0 < p^* \), followed by a succession of very small disclosures, leading the Firm’s belief gradually up all the way to one. This is illustrated in Figure 3 below.

Here is an alternative, heuristic derivation of the formula in (4). Note that, for \( p \geq p^* \), the payment that the type-1 Agent can extract from the Firm if the following posterior belief is \( p' \in \{0, p + dp\} \) is

\[
\frac{p}{p + dp}((p + dp) - (1 - \gamma(p + dp))) - (p - \gamma(1 - p)) = \gamma \frac{dp}{p} + o(dp).
\]

If the entire interval \([p^*, 1]\) is divided in this fashion, the resulting payoff tends then to

\[
\int_{p^*}^1 \gamma \frac{dp}{p} = \gamma (\ln 1 - \ln p^*) = -\gamma \ln p^*,
\]

which is the value that we found. This also illustrates that the limiting payoff is independent of the exact way that the information is divided up over time, as long as the mesh of the partition
tends to zero. The following corollary records the limiting value.

**Corollary 1** As $K \to \infty$, the optimal payoff tends to, for $p_0 < p^*$:

$$V_{1,\infty}(p_0) = -\gamma \ln p^*.$$ 

Note that this payoff is independent of $p_0$ (for $p_0 < p^*$), and indeed the first chunk of information, leading to a posterior belief of $p^*$, is given away for free, as it does not affect the Firm’s outside option. All later, very small releases of information do affect this outside option, and are priced accordingly.

Figure 4 illustrates how the payoff to the type-1 Agent varies with the number of rounds and the prior belief.

### 4.4 More General Payoffs

How did our results depend on our assumptions on the outside option? While the piecewise linear structure of the Firm’s payoff proves quite convenient for explicit formulas, the main results of subsection 4.3 generalize to more general payoff functions. Suppose so that the payoff of the Firm of its posterior belief $p$ after the $K$ rounds, is a non-decreasing function $w(\cdot)$ (we normalize $w(0) = 0$). This payoff can be thought as the reduced-form of some decision problem the firm faces, as in our baseline model. Then it is again in the interest of the type-1 Agent to split
information as finely as possible for any prior belief $p_0$ if and only if the average $w(p)/p$ is a strictly increasing function of $p$. If in a given round the Firm’s belief goes from $p$ to $p + dp$ (or 0, if case occurs), the Agent can charge up to

$$\frac{p}{p + dp} w(p + dp) - w(p) = (w'(p) - w(p)/p)dp + o(dp^2)$$

for it. Given the Firm’s prior belief $p_0$, the type-1 Agent’s payoff becomes then (in the limit, as the number of rounds $K$ goes to infinity)

$$\int_{p_0}^1 [w'(p) - w(p)/p]dp = w(1) - w(p_0) - \int_{p_0}^1 w(p)dp/p,$$

which generalizes the formula that we have seen for the special case $w(p) = (p - (1 - p)\gamma)^+$. That is, the type-1 Agent’s payoff is the area between the marginal payoff of the Firm and its average payoff. To see that this splitting information as finely as possible is optimal, consider
some arbitrary interval of beliefs \([p, \bar{p}]\). By having the posterior belief of the Firm jump from \(p\) to \(\bar{p}\), the payoff in that round is given by

\[
\frac{p}{\bar{p}} w(\bar{p}) - w(p).
\]

If instead this interval of beliefs is split as finely as is possible, the payoff over this range is

\[
w(\bar{p}) - w(p) - \int_p^{\bar{p}} \frac{w(p)}{p} dp.
\]

Hence, splitting is better if and only if

\[
\frac{1}{\bar{p} - p} \int_p^{\bar{p}} \frac{w(p)}{p} dp \leq \frac{w(\bar{p})}{\bar{p}},
\]

which is satisfied if the average \(w(p)/p\) is increasing. Conversely, if the average is decreasing over some range \([p, \bar{p}]\), then this argument shows that it is better to have the belief jump from \(p\) to \(\bar{p}\) than to split it as finely as possible.\(^\text{11}\) If the average is decreasing over some range, what determines the jump? Note that, as mentioned, the payoff from a jump is \(\frac{p w(\bar{p})}{\bar{p}} - w(p)\), while the marginal benefit from finely splitting at any given belief \(p\) (in particular, at \(\bar{p}\) and \(p\)) is \(w'(p) - w(p)/p\). Setting the marginal benefits equal at \(\bar{p}\) and \(p\) respectively yields that we must have

\[
\frac{w(\bar{p})}{\bar{p}} = \frac{w(p)}{p} \text{ and } w'(\bar{p}) = \frac{w(\bar{p})}{\bar{p}}.
\]

See Figure 5. The left panel illustrates how having two rounds improves on one round. Starting with a prior belief \(p_0\), the highest payoff the type-1 Agent can receive in one round is given by the dotted black segment. If instead information is disclosed in two steps, with an intermediary belief \(p_1\), the type-1 Agent’s payoff becomes the sum of the two red segments, which is strictly more, since \(w(p)/p\) is strictly increasing. The right panel illustrates the jump in beliefs that occurs over the relevant interval when the function is not strictly increasing, as occurs in our

\(^{11}\)If the average is constant over some interval (as in our example over the range \([0, p^*]\)), then in the limit it is irrelevant whether the belief jumps or not, but the limiting procedure we adopted will exclude any splitting over this range, since for any finite \(K\), such a split would correspond to a “wasted” round.
Figure 5: Splitting information with an arbitrary outside option

leading example for $p < p^*$.  

Another extreme feature of our model is that how much information the type-0 Agent owns is irrelevant to the Firm’s decision. In some cases, this seems like a reasonable assumption: having interesting ideas about how to build an electric bulb do not amount to much if some elements are missing. On the other hand, presenting some partial evidence that a rogue regime has weapons of mass destruction might affect the decision-maker’s opinion, and hence, his decision, even when it is common knowledge that this evidence is incomplete.

To model this, we may assume that, if the Firm’s belief is $p$, but the Agent fails to disclose any further evidence the posterior belief falls to $\lambda(p)$, where $\lambda$ is some non-creasing function. We have focused our attention on the case in which $\lambda(p) = 0$ for all $p < 1$. We could as well have picked an arbitrary function $\lambda$. In that case, following the reasoning detailed in the previous paragraphs, splitting information over some range of beliefs $[\underline{p}, \bar{p}]$ would lead to revenues

$$w(\bar{p}) - w(\underline{p}) - \int_\underline{p}^\bar{p} \frac{w(p) - w(\lambda(p))}{p - \lambda(p)} dp,$$
while jumping from belief $p$ to belief $\bar{p}$ in one round would yield
\[
w(\bar{p}) - w(p) - (\bar{p} - p) \frac{w(\bar{p}) - w(\lambda(\bar{p}))}{\bar{p} - \lambda(\bar{p})}.
\]
Hence, it is clear that splitting information is desirable if and only if the function \((w(p) - w(\lambda(p)))/(p - \lambda(p))\) is strictly increasing. Note that this is automatically satisfied if $w$ is strictly convex, independently of the function $\lambda$.

5 Inducing Effort

We now provide an example of motivation why equilibrium that maximizes $V_1$ may be of special interest if the selling information game is embedded into a larger game. In particular, suppose that before the selling game studied so far, there is an effort phase. In the effort phase the Agent chooses privately effort to induce probability of the state being 1, $p \in [0, \bar{p}]$. The cost of effort is $c(p)$ which is continuous and satisfies $c(0) = 0$, $c'(0) = 0$, $c'' > 0$. After choosing effort the Agent learns the state.

Total expected surplus in the game (with efficient decisions by the Firm) is
\[
p - c(p)
\]
Let $p_{FB}$ be the unique maximizer of it ($p_{FB} = c'^{-1}(1)$). Note that the surplus is increasing in $p$ for all $p < p_{FB}$.

Suppose the effort phase is followed by the selling information phase with number of rounds $K \to \infty$.

In equilibrium, if the Firm believes the Agent chooses $p$ and the Agent deviates to $\hat{p}$, the Agent expects payoff
\[
\hat{p}V_1(p) + (1 - \hat{p})V_0(p) - c(\hat{p})
\]
For any $p$, the optimal effort choice solves
\[
c'(\hat{p}) = V_1(p) - V_0(p)
\]
which is less than $p_{FB}$ since $V_1(p) - V_0(p) < 1$. In equilibrium of the over-all game we must have $\hat{p} = p$, so for $p$ to be an equilibrium we must have

$$c'(p) = V_1(p) - V_0(p)$$

What is the highest $p$ that is sustainable in equilibrium? Note that if we maximize $V_1(p) - V_0(p)$ for every $p$ then every $p$ such that

$$c'(p) \leq \max_p \{V_1(p) - V_0(p)\}$$

can be supported in equilibrium in which at the beginning of the selling information stage with some probability the players switch to the best equilibrium for the Firm (in which the Agent gets nothing). Hence, the largest $p$ consistent with equilibrium is supported by an equilibrium that maximizes $V_1(p) - V_0(p)$, which by part 5 of Lemma 1 is maximized by the equilibrium that maximizes $V_1(p)$ that we constructed above.

Of course, the equilibrium we constructed is not going to be unique in terms of maximizing incentives to exert effort in the whole game. Since in our equilibrium $V_0(p) > 0$, there exists a whole range of equilibria in which $V(p) - V_N(p)$ is the same as in our equilibrium, but that differ in the payoff achieved by the Firm. In particular, any equilibrium in which at the beginning of the communication stage the Agent pays the firm any $X \in [0, \overline{X}]$ is an equilibrium with the same difference $V_0(p) - V_1(p)$, where the bound:

$$\overline{X} = \frac{p - (p + \gamma (1 - p))^+ + \gamma p \min \{\ln p^*, \ln p\}}{1 - p}$$

is the payoff of the “Not Invest” type in the equilibrium we constructed above.

Finally, note that since $V_1(p) - V_0(p) < 1$, that weak information property rights (that would prevent the players from signing a contract with payments contingent on the released information) lead to inefficient effort event in the most efficient equilibria.
6 Intermediaries

6.1 The role of an intermediary

We have seen how the type-1 Agent, by selling information progressively, can obtain a payoff above the expected value $p_0$. Indeed, for $p_0 < p^*$, with sufficiently many rounds, this payoff approaches $-\gamma \ln p^*$. Nevertheless, this still falls short of the full value of information in this event. Without it, the Firm obtains a payoff of zero. With it, the Firm would obtain a payoff of one. Therefore, the actual value of information is one, but of course the Firm does not know it, and if it did, it would no longer be willing to pay for it. Could more general mechanisms help the type-1 Agent appropriate this value?

Note that, so far, the posterior belief of the Firm was either above its prior belief, or equal to zero. This is a consequence of the focus on pure strategies. If, in equilibrium, the type-1 Agent randomized over disclosing one or the other piece of information, the Firm’s posterior belief, conditional on seeing one of these pieces, could fall short of the prior belief, yet be strictly positive. For this to happen, it would suffice that the type-1 Agent is sufficiently more likely to disclose the other piece of information.

Yet, as we have already mentioned, there is no mixed-strategy equilibrium achieving a higher payoff to the type-1 Agent. Very roughly, this is because she would have to be indifferent over revealing either piece of information, and so her payoff could not exceed the one that she would obtain from disclosing the piece of information leading to a higher posterior belief.\(^{12}\)

Suppose now that the Agent has access to a trusted, disinterested intermediary, or a trusted computer. This intermediary, upon receiving the information $\iota$, could decide to randomize between releasing this piece of information or not. Therefore, if the piece of information is not disclosed, the Firm cannot draw a definite conclusion from this. Is this because the Agent failed to possess this information, or because it was censored by the intermediary? In this case, the Firm’s posterior belief could fall below its prior belief, yet remain positive. Note that the Agent may not be able to perform this garbling herself, because she need not be indifferent over the two resulting outcomes.

\(^{12}\)Introducing a public randomization would not help either, since making the randomization observable only helps to convexify the equilibrium payoff set, which is of no help here.
While we shall show that this elementary garbling, leading to only two possible posterior beliefs, is sufficient to achieve the best equilibrium asymptotically, one can conceive of more complicated garblings. For instance, the Agent might be expected to transmit two pieces of information to the intermediary, and, depending upon which pieces of information he obtains, the intermediary could randomize over disclosing none, one or the other, or both of these pieces of information. This would lead to four possible posterior beliefs.

6.2 An Illustration

Consider the simple example in which \( \gamma = 1 \), so that \( p^* = 1/2 \). The right panel of Figure 6 illustrates one of the procedures that the intermediary may follow, starting from a given belief \( p_0 = 1/3 \). Here, the intermediary sends one of two messages, low or high. The high message makes the Firm more optimistic, with a corresponding posterior of 1/2. The low message makes the Firm more pessimistic, with a posterior of 1/6. Because the Firm’s belief is a martingale, and because \( p_0 = 1/3 \) is the mean of 1/2 and 1/6, the two messages must be equally likely from the Firm’s point of view.

How likely is each message from the type-1 Agent’s point of view? Note that the low posterior, 1/6, is half as high as the prior belief, 1/3. This means that, from the Firm’s point of view, the low messages is half likely to be observed when the state is \( \omega_1 \), i.e. when the Agent is of type 1, as the high message. Because he assigns an unconditional probability of 1/2 to the low message, he must then assign probability \( 1/2 \cdot 1/2 = 1/4 \) to this low message conditional on the Agent being of type 1. This is then the probability that the type-1 Agent must assign to this low message.

The left panel of Figure 6 depicts the three continuation payoffs in the best equilibrium characterized in the previous section, without an intermediary, starting from a belief 1/6. The type-1 Agent receives \( -\gamma \ln p^* = \ln 2 \), the Firm receives \( w(1/6) = 0 \), yet the sum of all three payoffs must equal the surplus \( p = 1/6 \), so that the type-0’s Agent payoff can be read off the y-axis as shown. Note that \( 0 < V_0 \leq V_1 \).

Consider then the following scheme. In the second stage, it is understood that the Agent will make a payment of \( V_0 \) to the Firm if and only if the realized message is low (in particular,
there is no payment by the Agent to the Firm if the realized message is high). Aside from this one-time, conditional payment from the Agent to the Firm, all payments by the Firm to the Agent, and all information disclosures from the Agent to the Firm occur from the second round onwards as in the equilibrium without an intermediary (which is obviously possible even with an intermediary), given the realized message.

In the initial round, before the message is sent, the Firm must pay the difference between his expected continuation payoff and his current outside option, 0. If he fails to do so, no further messages are sent. How much is the Firm willing to pay? Note that, if there was no payment from the Agent to the Firm conditional on a low message, he is not willing to pay anything, since his outside option after either message is still 0. Nevertheless, because he expects to receive $V_0$ in an event whose probability is $1/2$ from his point of view, he is willing to pay up to $V_0/2$ upfront in this scheme. How much is this scheme worth to the type-1 Agent? Her expected payoff is

$$\frac{1}{2}V_0 + \frac{1}{4}(V_1 - V_0) + \frac{3}{4}V_1 = V_1 + \frac{1}{4}V_0 > V_1.$$  

To see this, note that she gets $V_0/2$ upfront, $V_1 - V_0$ in the event that the message is low (an event to which she assigns probability $1/4$) and $V_1$ in the event that the message is high. As a
result, with this scheme, her payoff with a prior 1/3 is strictly larger than $V_1$, which was what it is equal to without an intermediary.

Observe first that such a scheme is not possible without an intermediary, because the type-1 Agent is not indifferent over realized messages. She strictly prefers the high message to obtain, so that such a scheme cannot be replicated by mixed strategies without an intermediary. Second, note that the payment that the Agent makes if a low message occurs is not informative per se. This is because this payment is no larger than $V_0$, and the continuation payoffs of the Agent is at least as much independently of her type. Higher payments would not work, because the type-0 Agent would not be willing to make it given the continuation equilibrium, and so the occurrence of a payment or not would transmit information about the Agent’s type. From the left panel of Figure 6, it is clear that, the closer the expected payoff $pV_1$ of the type-1 Agent is to the total surplus $p$, the smaller is the resulting $V_0$, and so, the smaller the scope for such a scheme becomes. But as long as $V_0$ remains strictly positive, such schemes remain possible.

This scheme is nothing but a bet, or a trade, between two agents whose beliefs about some event disagree. The Firm attaches probability 1/4 to the event that the posterior will be 1/4, while the Firm attaches probability 1/2 to this event. Therefore, there is room for a profitable trade, and the only bound on this trade is that the bet cannot exceed the type-0’s continuation payoff. Note that the type-0 Agent loses from this scheme, for she is the one who assigns a high probability to the even that the posterior is 1/4.

It is then clear how an intermediary might help. Without an intermediary, it was possible to drive the Firm’s payoff to his outside option. With an intermediary, we can further drive the type-0 Agent’s payoff down to her outside option, namely zero. In this fashion, the intermediary might help the type-1 Agent to extract the full surplus. In the next subsection, we shall see that this is precisely the case.

### 6.3 The Optimization Problem

In what follows, there is not much to gain from sticking with our leading example. The outside option $w$ of the Firm, then, is an arbitrary continuous function with $w(0) = 0, w(1) = 1$, with the feature that full information is efficient, i.e. $w(p) \leq p$ for all $p \in [0, 1]$. 
The procedure used by the intermediary can be summarized by a distribution $F_k(\cdot|p)$ over the Firm’s posterior beliefs, given the prior belief $p$, and given the number of rounds $k$. Because this distribution is known, the Firm’s belief must be a martingale, which means that, given $p$,

$$
\int_0^1 p' dF_k(p'|p) = p, \text{ or } \int_0^1 (p' - p) dF_k(p'|p) = 0.
$$

(5)

To put it differently, $F(\cdot|p)$ is a mean-preserving spread of the Firm’s prior belief $p$.

Given such a distribution, and some equilibrium to be played in the continuation game for each resulting posterior belief $p'$, how much is the Firm willing to pay upfront? Again, this must be the difference between its continuation payoff and its outside option, namely

$$\tilde{t}_k^F := \int_0^1 (w(p') + X(p')) dF_k(p'|p) - w(p).$$

where, as before, $X(p')$, or $X'$ for short, denotes the Firm’s payoff, net of its outside option, in the continuation game given that the posterior belief is $p'$.

Assume that the distribution $F(\cdot|p)$ assigns probability $q$ to some posterior belief $p'$. This means that the Firm attaches probability $q$ to its posterior belief being $p'$. What is the probability $q_1$ assigned to this event by the type-1 Agent? This must be $qp'/p$, because

$$p' = Pr[\omega = 1|p'] = \frac{pq_1}{q},$$

where the first equality from the definition of the event $p'$, and the second follows from Bayes’ rule, given the prior belief $p$. This is an obvious generalization of the calculation that we have seen in the simple example of subsection 6.2.

Therefore, the maximal payoff that the type-1 Agent expects to receive from the next round onward is

$$\int_0^1 V_{1,k-1}(p', X') \frac{p'}{p} dF_k(p'|p),$$

---

13One interpretation of $F_k$ is that the Agent reveals fully the state of the world to the intermediary and then in every round the intermediary releases a noisy signal correlated with the true state of the world in such a way that the distribution of posteriors is $F_k$. 33
where, as before, \( V_{1,k-1}(p', X') \) denotes the maximal payoff of the type-1 Agent, with \( k-1 \) rounds to go, given that the Firm’s net payoff is \( X' \), and its belief is \( p' \).

Combining these two observations, we obtain that the payoff of the type-1 Agent is at most

\[
\int_0^1 (w(p') + X(p')) dF_k(p'|p) - w(p) + \int_0^1 V_{1,k-1}(p', X') \frac{p'}{p} dF_k(p'|p),
\]

and our objective is to maximize this expression over all distributions \( F_k(\cdot|p) \), as well as mappings \( p' \mapsto X(p') \) (subject to (5)).

### 6.4 The optimal transfers

This requires, first of all, an understanding of the function \( V_{1,k}(p, X) \). Note that, as observed earlier, we can always assume that the equilibrium is efficient: take any equilibrium, and assume that, in the last round, on the equilibrium path, the type-1 Agent discloses the state. This modification can only relax any incentive (or individual rationality) constraint. This means that payoffs must satisfy (3), which provides a rather elementary upper bound on the maximal payoff to the type-1 Agent: in the best possible case, the payoffs \( X \) and \( V_{0,k}(p, X) \) are zero, and hence we have

\[
V_{1,k}(p) \leq p - w(p).
\]

Our observation from Lemma 1 that the equilibrium that maximizes the type-1 Agent’s payoff also maximizes the sum of the Firm’s and type-1 Agent’s payoffs is obviously true here as well. Hence, any increase in \( X \) must lead to a decrease in \( V_{1,k}(p, X) \) of at least that amount. As long as \( X \) is such that \( V_{0,k}(p, X) \) is positive, we do not need to decrease \( V_{1,k}(p, X) \) by more than this amount, because it is then possible to simply decrease the net transfer made by the Firm to the Agent in the initial period by as much. Therefore, either \( V_{1,k}(p, X) = V_{1,k}(p) - X \), if \( X \) is smaller than some threshold value \( X^*_k(p) \) (\( X^* \) for short), or \( V_{0,k}(p, X) = 0 \). By continuity, it must be that, at \( X = X^* \),

\[
p(V_{1,k}(p) - X^*) + X^* + w(p) = p, \text{ or } X^* = \frac{p(1 - V_{1,k}(p)) - w(p)}{1 - p}.
\]
Therefore, for values of $X$ below $X^*$, we have that $V_1(p, X) = V_{1,k}(p) - X$, and this payoff is obtained from the equilibrium achieving the payoff $V_{1,k}(p)$ to the type-1 Agent, by reducing the net transfer from the Firm to the Agent in the initial round by an amount $X$. For values of $X$ above $X^*$, we know that $V_{0,k}(p, X) = 0$, so that

$$V_{1,k}(p, X) \leq 1 - \frac{w(p) + X}{p}.$$ 

We may now turn to the issue of the optimal net payoff to grant the Firm in the continuation round. This can be done pointwise, for each posterior belief $p'$. The previous analysis suggests that, to identify what the optimal value of $X'$ is, it is convenient to break down the analysis into two cases, according to whether or not $X'$ is above $X^*$. Consider some posterior belief $p'$ in the support of the distribution $F_k(\cdot|p)$. From (6), the contribution to the type-1 Agent's payoff from this posterior is equal to

$$w(p') + X' + V_{1,k-1}(p', X') \frac{p'}{p} \begin{cases} = w(p') + X' + (V_{1,k-1}(p') - X') \frac{p'}{p} & \text{if } X' \leq X^*(p'), \\ \leq w(p') + X' + \left(1 - \frac{w(p') + X'}{p'}\right) \frac{p'}{p} & \text{if } X' > X^*(p'). \end{cases}$$

Note that, for $X' > X^*(p')$, the upper bound to this contribution is decreasing in $X'$, and since this upper bound is achieved at $X' = X^*(p')$, it is best to set $X' = X^*(p')$ in this range. For $X' \leq X^*(p')$, this depends on $p'$: if $p' > p$, it is best to set $X'$ to zero, while if $p' < p$, it is optimal to set $X'$ to $X^*(p')$. To conclude, the optimal choice of $X'$ is

$$X(p') = \begin{cases} X^*(p') & \text{if } p' < p, \\ 0 & \text{if } p' \geq p. \end{cases}$$

The key intuition here is that the “Invest” type assigns a higher probability to the event that the posterior will be $p' > p$ than does the Firm and conversely, a lower probability to the event that $p' < p$, because she knows that the state is 1. Therefore, the “Invest” type wants to offer the Firm extra continuation payoff in the event that $p' < p$ (and collect extra money for it now), and offer as little continuation payoff as possible in the event that $p' > p$. Since the Agent and
the Firm have different beliefs, there is room for profitable bets, in the form of transfers whose odds are actuarially fair from the Firm’s point of view, but profitable from the point of view of the “Invest” Agent. Such bets were not possible without the intermediary, because, at the only posterior belief lower than \( p \), namely \( p' = 0 \), there was no room for any further transfer in this event (because there was no further information to be sold).

### 6.5 The value of an intermediary

Having solved for the optimal transfers, we may now focus on the issue of identifying the optimal distribution \( F_k(\cdot|p) \). Plugging in our solution for \( X' \) into (6), we obtain that

\[
V_{1,k}(p) = \sup_{F_k(\cdot|p)} \int_0^1 v_{k-1}(p'; p) dF_k (p'|p) - w(p),
\]

where

\[
v_{k-1}(p'; p) := \begin{cases} 
  w(p') + \frac{p'}{p} X^*(p') + \frac{p'}{p} V_{1,k-1}(p') & \text{for } p' < p, \\
  w(p') + \frac{p'}{p} V_{1,k-1}(p') & \text{for } p' \geq p,
\end{cases}
\]

and the supremum is taken over all distributions \( F_k(\cdot|p) \) that satisfy (5), namely, \( F_k(\cdot|p) \) must be a distribution with mean \( p \).

This optimality equation cannot be solved explicitly. Nevertheless, the associated operator is monotone, and bounded above. Therefore, its limiting value as we let \( k \) tend to infinity, using the initial value \( V_{1,0}(p) = 0 \) for all \( p \), converges to the smallest (positive) fixed point of this operator. This fixed point gives us the limiting payoff of the “Invest” type as the number of rounds grows without bound.

It turns out that we can guess this fixed point. One of the fixed points of (7) is

\[
V_1(p) = p - w(p)
\]

Recall that this value is the upper bound on \( V_{1,k}(p) \) we had derived earlier, so it is the highest payoff that we could have hoped for. In fact, we have:
Figure 7: Payoffs with or without an intermediary ($\gamma = 1$, $V_{\text{int}}$ is the payoff with an intermediary)

Theorem 2  The function defined by, for all $p$,

$$V_{1}^{\text{int}}(p) := p - w(p)$$

is the limiting value of the game to the “Invest” type, as the number of rounds tends to infinity.

Proof. See appendix. □

Figure 7 provides a side-by-side comparison between the payoff to the type-1 Agent in the best equilibrium with and without intermediary. In this case, this value is

$$V_{1}^{\text{int}}(p) = \begin{cases} 1 & \text{for } p \leq p^*, \\ \gamma \frac{1-p}{p} & \text{for } p \geq p^*. \end{cases}$$

As is clear from the proof, the intermediary achieves full extraction through bets similar to the one in subsection 6.2 and very small changes in the Firm’s belief. That is, from any belief
p onward, it assigns equal probability to the posterior beliefs $p - \varepsilon$ and $p - \varepsilon$, for some small $\varepsilon > 0$, accompanied by an upfront payment by the Firm that includes a maximal payback by the Agent to the Firm in case the posterior belief is low. The maximal payback is determined by the continuation value of the type-0 Agent. Full extraction obtains then asymptotically as $\varepsilon$ tends to 0, and the number of rounds tends to infinity. Of course, it is not the only way that the intermediary can achieve this maximum. In our leading example, for instance, in which $w(p) = (p - (1 - p)\gamma)^+$, the posterior beliefs when $p > p^*$ can be chosen to be either (arbitrarily close to) $p^*$ or 1.

7 Final Remarks

The model can be extended in many ways. For example:

1. Suppose that there is discounting with every round of communication. Then it will no longer be true that adding a round of communication will be always beneficial to the Agent, because there will be a trade-off between collecting more money and collecting it earlier and hence the optimal number of rounds used in equilibrium will be finite. However, for high enough discount factors the optimal equilibrium is still going to involve a gradual release of information.

2. Suppose that the Agent cares a little (say $\varepsilon$) that the Firm takes the correct action. Then in the finite game without discounting there is going to be a unique equilibrium: the Firm would never pay the Agent since it knows that in the last round it is a dominant strategy for the Agent to reveal the correct action. However, in an infinite-horizon game with discounting we would not have such unraveling and it is possible to sustain equilibria in which the Agent is paid for a gradual release of information (however, $\varepsilon$ will restrict how close to an extreme posterior the Agent can get in any equilibrium without being compelled to immediately release all remaining information).

3. Another direction is to consider a situation in which the Agent has information about an invention and can release parts of it slowly (and collect money for the partial information). The difference from our current setup is that in that case the Firm would not know what action to take until it learns the whole invention (i.e. the Firm would not know how to “Invest” in the
invention on its own). In the simplest model the solution would be simple: the Agent should reveal to the Firm all but a small detail of the invention, increasing the Firm’s belief $p$ that the invention is valuable as much as possible, and then sell the information for $p$. (That strategy does not work our model since once Firm’s belief is high enough, the Firm can get a high payoff by investing without any further information.) A more interesting model would allow the Firm to know a priori and privately some of the elements of the invention: in that case an Agent revealing all but a small elements would risk that the Firm would know already the missing piece. In that case we conjecture that again it would be optimal for the successful inventor to release information more gradually.

References


Appendix

A. Proof of Proposition 3

We provide here the missing details for the proof of Proposition 3. In Section 3, the result has been established for pure strategies. What remains to be shown is that there exists no mixed-strategy equilibrium yielding a higher payoff $V_1$. Consider then a candidate mixed-strategy equilibrium. This strategy profile can be summarized by a distribution $F_{k+1}(\cdot|p)$ that is used by the Agent (on the equilibrium path) with $k + 1$ rounds left, given belief $p$, and the continuation payoffs $W_k(\cdot)$, and $V_k(\cdot)$. As before, we may assume that the equilibrium is efficient, and so we can assume that, given that the Firm obtains a net payoff of $X_k$ (i.e., given that $W_k = (p - (1 - p)\gamma)^+ + X_k$), the type-1 Agent receives $V_{1,k}(p, X_k)$, the highest payoff to this type given that the Firm receives at least a net payoff of $X_k$. Since $V_{1,k}$ maximizes the sum of the Firm’s and type-1 Agent’s payoff (see Lemma 1), we have that, for all $k, p$ and $X \geq 0$,

$$V_{1,k}(p, X) \leq V_{1,k}(p) - X.$$ 

The payoff $V_{1,k+1}(p)$ of the type-1 Agent is at most, with $k + 1$ rounds to go,

$$\sup_{F_{k+1}(\cdot|p)} \int_0^1 \left[ (p' - (1 - p')\gamma)^+ + X_k(p') + V_{1,k}(p', X_k(p')) \frac{p'}{p} \right] dF_{k+1}(p'|p) - (p - (1 - p)\gamma)^+,$$

where the supremum is taken over all distributions $F_{k+1}(\cdot|p)$ that satisfy,

$$\int_{[0,1]} (p' - p)dF_{k+1}(p'|p) = 0$$

i.e. that the belief of the Firm must follow a martingale. To emphasize the importance of the posterior $p' = 0$, we alternatively write this constraint as $\int_0^1 (p' - p)dF_{k+1}(p'|p) = pF_{k+1}(0|p),$ where $\int_0^1 dF_{k+1}(p'|p) = 1 - F_{k+1}(0|p)$.

Mixing by the type-1 Agent requires indifference, that is, for all $p' > 0$ in the support of $F_{k+1}(\cdot|p)$, $V_{1,k}(p', X') = V_k$ for some $V_k$ independent of $p'$. (The existence of such a mixed-strategy equilibrium would also impose additional restrictions, but since our purpose is to show
that such an equilibrium cannot improve upon the pure-strategy equilibrium, we disregard them here.) Relaxing possibly the constraints (which can only increase the value of the objective), assume then that, for all such values of \( p' \), and \( X \geq 0 \),

\[
V_{1,k}(p', X) = V_{1,k}(p') - X.
\]

By substitution, we obtain that \( V_{1,k+1}(p) \) is at most equal to, for some \( V_k \),

\[
\sup_{F_{k+1}(\cdot|p)} \int_0^1 [(p' - (1 - p')\gamma)^+ + V_{1,k}(p') - V_k + \frac{p'}{p}] dF_{k+1}(p'|p) - (p - (1 - p)\gamma)^+ = \sup_{F_{k+1}(\cdot|p)} \int_0^1 [(p' - (1 - p')\gamma)^+ + V_{1,k}(p')] dF_{k+1}(p'|p) + F_{k+1}(0|p)V_k - (p - (1 - p)\gamma)^+.
\]

Because \( V_k \leq V_{1,k}(p') \) for all \( p' > 0 \) in the support of \( F_{k+1}(\cdot|p) \), it is then clear that this is largest when \( V_k = \min_{p' \in \text{supp } F_{k+1}(\cdot|p)} V_{1,k}(p') \). We claim that, along with the structure of the continuation payoff, this observation is sufficient to ensure that the optimal distribution \( F_{k+1}(\cdot|p) \) assigns positive probability to only two posterior beliefs, namely 0 and some \( p' > p \). This will establish that attention can be restricted to pure strategies.

The claim is proved by induction. With only round left, the result is obvious, because the Firm cannot pay more than \( p - (p - (1 - p)\gamma)^+ \), and this is what the Firm pays in the pure-strategy equilibrium.

Assume that, defining \( Z_k \) as \( Z_k(p) = V_{1,k}(p) + (p - (1 - p)\gamma)^+ - p \), all \( p \), we have

\[
Z_k(p) = V_{1,k-1}(p^*) - p \quad \text{if} \quad p < p_k := p^{\frac{1}{k+1}}, \quad \text{and} \quad Z_k(p) = \gamma[k(1 - p^{1/k}) - (1 - p)] \quad \text{otherwise}.
\]

These are the payoffs with \( k \) rounds to go, in the pure-strategy equilibrium, and our induction hypothesis is that these are also the highest equilibrium payoffs. Given our earlier observation,

42
Let us define \( x \) or re-arranging, we get the expression can assign positive probability to only one strictly positive value of \( W \). Note that the function \( k \) equilibrium with the highest payoff to the type-1 Agent with the desired formula for the payoff \( Z \) strategy, and it must be the one identified in the proof of Proposition 3, in the text (which gives the desired formula for the payoff \( Z_{k+1} \)). \[ \square \]

**B. Proof of Theorem 2**

Recall that the function to be maximized is

\[
\frac{\int_0^p [w(p') + V_{1,k-1}(p')] p'}{p} + \frac{p - p' p (1 - V_{1,k-1}(p')) - w(p')}{1 - p'} |dF_k(p'|p)|
\]

or re-arranging,

\[
\int_0^p \left( \frac{1 - p}{p(1 - p')} (p' w(p') + p' V_{1,k-1}(p')) + \frac{(p - p') p'}{p(1 - p')} \right) dF_k(p'|p) + \int_1^p [w(p') + V_{1,k-1}(p')] p' |dF_k(p'|p)| - w(p),
\]

Let us define \( x_k(p) := p - w(p) - p V_{1,k}(p) \), and so multiplying through by \( p \), and substituting, we get

\[
p - w(p) - x_k(p) = \int_0^p \left[ \frac{1 - p}{1 - p'} (p' w(p') + p' - w(p') - x_{k-1}(p')) + \frac{(p - p') p'}{1 - p'} \right] dF_k(p'|p)
\]

\[
+ \int_1^p [pw(p') + p' - w(p') - x_{k-1}(p')] dF_k(p'|p) - pw(p),
\]

\[ 43 \]
or re-arranging,

\[ x_k(p) = p - w(p) - \int_0^p \left( \frac{1-p}{1-p'} (p' - 1)w(p') - x_{k-1}(p') \right) + p'|dF_k(p'|p) \]

\[- \int_p^1 [p' - (1-p)w(p') - x_{k-1}(p')]dF_k(p'|p) + pw(p), \]

or more one time,

\[ x_k(p) = (1-p) \int_0^p \frac{x_{k-1}(p')}{1-p'}dF_k(p'|p) + \int_p^1 x_{k-1}(p')dF_k(p'|p) + (1-p) \int_0^1 (w(p') - w(p))dF_k(p'|p). \]

Note that the operator mapping \( x_{k-1} \) into \( x_k \), as defined by the minimum over \( F_k(\cdot|p) \) for each \( p \) is a monotone operator. Note also that \( x = 0 \) is a fixed point of this operator (consider \( F_k(\cdot|p) = \delta_p \), the Dirac measure at \( p \)). We therefore ask whether this operator admits a larger fixed point. So we consider the optimality equation, which to each \( p \) associates

\[ x(p) = \min_{F(\cdot|p)} \{(1-p) \int_0^p \frac{x(p')}{1-p'}dF(p'|p) + \int_p^1 x(p')dF(p'|p) + (1-p) \int_0^1 (w(p') - w(p))dF(p'|p) \}. \]

It is standard to show that \( x \) must be continuous on \((0,1)\). Further, consider the feasible \( F(\cdot|p) \) is the distribution that assigns probability \( 1/2 \) to \( p - \varepsilon \), and \( 1/2 \) to \( p + \varepsilon \), for \( \varepsilon > 0 \) small enough. This implies that

\[ x(p) \leq \frac{1}{2} \frac{1-p}{1-p + \varepsilon} x(p - \varepsilon) + \frac{1}{2} x(p + \varepsilon) + (1-p) \left( \frac{w(p + \varepsilon) + w(p - \varepsilon)}{2} - w(p) \right), \quad (8) \]

or

\[ x(p) + (1-p)w(p) \leq \frac{1}{2} \frac{1-p}{1-p + \varepsilon} (x(p - \varepsilon) + (1-p + \varepsilon)w(p - \varepsilon)) \]

\[ + \frac{1}{2} (x(p + \varepsilon) + (1-p - \varepsilon)w(p + \varepsilon)) + \varepsilon w(p + \varepsilon) \]

\[ = \frac{1}{2} (x(p - \varepsilon) + (1-p + \varepsilon)w(p - \varepsilon)) + \frac{1}{2} (x(p + \varepsilon) + (1-p - \varepsilon)w(p + \varepsilon)) \]

\[ + \varepsilon \left( w(p + \varepsilon) - w(p - \varepsilon) - \frac{x(p - \varepsilon)}{1-p + \varepsilon} \right). \]
Suppose that $x(p) > 0$ for some $p \in (0, 1)$. The, since $x$ is continuous, $x > 0$ on some interval $I$. Because $w$ is continuous, the last summand is then negative for all $p \in I$, for $\varepsilon > 0$ small enough. This implies that the function $z : p \mapsto x(p) + (1 - p)w(p)$ is convex on $I$, and therefore differentiable a.e. on $I$. Re-arranging our last inequality, we have

$$2 \left( w(p - \varepsilon) - w(p + \varepsilon) + \frac{x(p - \varepsilon)}{1 - p + \varepsilon} \right) + \frac{z(p) - z(p - \varepsilon)}{\varepsilon} \leq \frac{z(p + \varepsilon) - z(p)}{\varepsilon}.$$ 

Integrating over $I$, taking limits as $\varepsilon \to 0$ and using the a.e. differentiability of $z$ gives

$$\int_I \frac{x(p)}{1 - p} \leq 0,$$

Since $x \geq 0$ and is continuous, it follows that $x = 0$ on $I$. Since $I$ was arbitrary, it follows that $x = 0$ on $(0, 1)$.

Because $x$ is the largest fixed point of the optimality equations, and because the map defined by these equations is monotone, it follows that the limit of the iterations of these maps, starting with $x_0(p) = p - w(p) - pV_{1,0}(p)$ is well-defined and equal to 0. Given the definition of $x$, the claim regarding the limiting value of $V_{1,k}$ follows. □