Entrepreneurial Finance and Non-diversifiable Risk*

Hui Chen†  Jianjun Miao‡  Neng Wang§

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†MIT Sloan School of Management. Email: huichen@mit.edu. Tel.: 617-324-3896.
‡Department of Economics, Boston University. Email: miaoj@bu.edu. Tel.: 617-353-6675.
§Columbia Business School and NBER. Email: neng.wang@columbia.edu. Tel.: 212-854-3869.
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Abstract

Entrepreneurs face significant non-diversifiable business risks. We build a dynamic incomplete-markets model of entrepreneurial firms to demonstrate the important implications of non-diversifiable risks for entrepreneurs’ interdependent consumption, portfolio allocation, financing, investment, and business exit (cash-out and default) decisions. The optimal capital structure is determined by a generalized tradeoff model where leverage via risky non-recourse debt provides significant diversification benefits. More risk-averse entrepreneurs default earlier, but also choose higher leverage, even though leverage makes his equity more risky. Cash-out option and external equity further improve diversification and raise the entrepreneur’s valuation of the firm. Non-diversified entrepreneurs demand both systematic and idiosyncratic risk premia for holding the firm. We provide a tractable and operational framework to value equity and the nontradable default/cash-out options, and analytically characterize the idiosyncratic risk premium. Finally, entrepreneurial risk aversion can overturn the risk-shifting incentives induced by risky debt.

Keywords: Default, diversification benefits, entrepreneurial risk aversion, incomplete markets, private equity premium, hedging, capital structure, cash-out option, precautionary saving

JEL Classification: G11, G31, E2
1 Introduction

An entrepreneur is “one who organizes, manages, and assumes the risks of a business or enterprise” (Merriam-Webster). Indeed, lack of diversification is one of the defining characteristics of entrepreneurship. Numerous empirical studies have documented that (i) active businesses account for a large fraction of entrepreneurs’ total wealth (for example, Moskowitz and Vissing-Jorgensen (2002) find that about 75 percent of all private equity is owned by households for whom it constitutes at least half of their total net worth), and (ii) entrepreneurial firms tend to have highly concentrated ownership (for example, using data from the Survey of Small Business Finances, Heaton and Lucas (2004) document that the principal owner holds on average 81 percent of the firm’s equity, and the median owner wholly owns the firm). Moreover, economic theory based on conflicts of interest and informational asymmetry (between the entrepreneur and financiers) has provided the micro foundation for concentrated ownership.

Motivated by both the empirical evidence and the micro theory, we provide a first dynamic incomplete-markets model that explicitly incorporates the effect of non-diversifiable risk on the valuation and intertemporal decision making (investment, financing, business exit) for an entrepreneurial firm. We achieve this objective by unifying a workhorse dynamic corporate finance model (Leland (1994)) with the incomplete-markets consumption smoothing/precautionary saving literature (e.g. Friedman (1957), Hall (1978), Deaton (1991)) and consumption/portfolio choice models (e.g. Merton (1971)). We show that non-diversifiable business risk not only generates quantitatively significant effects on dynamic capital budgeting, financing, business exits, and valuation of entrepreneurial firms, but provides a range of novel empirical predictions. Our framework is also applicable to public firms with concentrated managerial ownership (e.g. those run by controlling shareholders and/or under-diversified managers).

Besides high concentration of equity ownership, another important feature of private businesses is that they predominantly rely on debt for outside funding (see Heaton and Lucas (2004), Robb

\footnote{Among the other empirical studies are Gentry and Hubbard (2004), Berger and Udell (1998), Cole and Wolken (1996), and Petersen and Rajan (1994).}

\footnote{Bitler, Moskowitz, and Vissing-Jorgensen (2005) provide evidence that agency considerations play a key role in explaining why entrepreneurs on average hold large ownership shares.}

\footnote{See earlier work of Black and Cox (1976), Fischer, Heinkel, and Zechner (1989), and recent developments of Goldstein, Ju, and Leland (2001), Strebulaev (2007), Hackbarth, Miao, and Morellec (2006), Bhamra, Kuehn, and Strebulaev (2009), and Chen (2008).}
and Robinson (2009) for US evidence, and Brav (2009) for UK evidence). This could be due to the fact that the small size of many private businesses does not justify the fixed costs for issuing outside equity. Strong information asymmetry and incentive alignment problems from a diluted ownership also tend to make debt dominate equity for these firms. As a result, the main components of financing for the majority of these firms are inside equity and outside debt.

What determines the optimal mix of inside equity and outside debt? Due to market incompleteness, the diversification benefits of risky debt becomes a key factor in addition to the standard tradeoff between tax benefits and costs of financial distress. Zame (1993) argues that risky debt has the advantage in helping complete the markets. Heaton and Lucas (2004) provide the first model of the diversification benefits of risky debt for entrepreneurial firms in a static setting, and analyze the interaction between capital budgeting, capital structure, and portfolio choice for the entrepreneur. Building on their insight, we extend the analysis to a dynamic setting, and incorporate business exit (cash-out), outside equity, project choice, and tax considerations for the entrepreneurial firm. These features not only make the model more realistic, but have important effects on firm decisions. For example, like default, the option to cash out also helps complete the market and can have large effects on firms’ financing choices. Moreover, we provide analytical characterization of capital budgeting/hurdle rate, capital structure tradeoff, and endogenous exit decisions.

We consider a risk-averse entrepreneur with access to an illiquid non-tradable investment project. The project requires a lump-sum investment to start up, and generates stochastic cash flows that bear both systematic and idiosyncratic risks. Like a consumer, he makes intertemporal consumption/saving decisions and allocates his liquid wealth between a riskless asset and a diversified market portfolio (as in Merton (1971)). Like a firm, the entrepreneur needs to make investment/capital budgeting, financing, and exit decisions.

If he chooses to take on the project, the entrepreneur sets up a firm with limited liability (e.g. limited liability companies (LLC) or an S corporation), which makes debt non-recourse. Moreover, the LLC or S corporation allows the entrepreneur to face single-layer taxation for his business income. In normal business times, the entrepreneur uses business income to service the firm’s debt.

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4Leland and Pyle (1977) and Myers and Majluf (1984) argue that debt often dominates equity in settings with asymmetric information because debt is less information-sensitive. Jensen and Meckling (1976) suggest that managers with low levels of ownership may exert less effort, which is arguably more important to the growth of private businesses than well-established public firms.
If the firm’s revenue falls short of servicing its debt, the entrepreneur may still find it optimal to use his personal savings to service the debt in order to continue to the firm’s operation. However, when revenue becomes sufficiently low, the entrepreneur defaults on the debt, which triggers inefficient liquidation as in classic tradeoff models of corporate finance. If the firm does sufficiently well, he might choose to incur the transaction and other costs (such as taxes), repay the debt in full, and realize the capital gains by selling the firm to cash out. After exiting from his business (cash-out or default), the entrepreneur becomes a regular household and lives only on his financial wealth. Cash-out and default allow the entrepreneur to achieve diversification benefits. These business exit decisions are essentially (non-tradable) American style options on the illiquid project and take the form of endogenous double-threshold policies.

Importantly, the entrepreneur’s business income and wealth accumulation are endogenously affected by the firm’s capital budgeting, leverage, and business exit decisions. While he can hedge the systematic component of his business risks using the market portfolio, he cannot diversify the idiosyncratic risks. Therefore, the entrepreneur faces incomplete markets, and the idiosyncratic risk exposure will affect his interdependent consumption, investment, financing, and business exit decisions. Such non-diversifiable idiosyncratic risk makes entrepreneurial finance distinct from the standard textbook treatment of corporate finance, and can sometimes overturn the predictions of standard finance theory on firm valuation, financing choices, and agency problems.

To summarize, our incomplete-markets model provides a framework to analyze the effect of nondiversifiable business risk on (i) the tradeoff between external financing and inside equity (ii) endogenous default option, (iii) endogenous cash-out option, (iv) consumption/saving, (v) portfolio choice/dynamic hedging, (vi) business investment, and (vii) valuation of the firm. While each decision taken in isolation is interesting, we show that the interactions among these different margins generate new economic insights that are essential to our understand of entrepreneurial finance.

The main results of the model are the following. First, on capital structure, our framework provides a generalized dynamic tradeoff model, where in addition to the standard tradeoff between tax benefits of debt and costs of financial distress/agency (as in Leland (1994)), risky debt also provides diversification benefits. This is because risky debt helps reduce the entrepreneur’s exposure

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5This one-time entrepreneurship assumption does not affect the model’s key economic mechanism (i.e. non-diversifiable risk) in any significant way. We leave extensions to allow for repeat entrepreneurship for future research.
to idiosyncratic business risk by enabling risk sharing in the default states. Hence, he rationally chooses more debt and hence higher leverage for the firm. The options of default and cash-out in our model have important feedback effects on the capital structure and pricing of credit risk. Our analysis also suggests that the natural measure of leverage for entrepreneurial firms is private leverage, defined as the ratio of public (market) value of debt and the private (subjective) value of firm. This private leverage captures the diversification benefits of risky debt and highlights the tradeoff between inside equity and outside debt.

The diversification benefits of debt are large. Even without any tax benefit of debt, the entrepreneurial firm still issues a significant amount of debt. The diversification benefits also lead to a seemingly counterintuitive prediction: more risk-averse entrepreneurs prefer higher leverage. On the one hand, higher leverage increases the risk of the entrepreneur’s equity stake within the firm. On the other hand, higher leverage implies less equity exposure to the entrepreneurial project, making the entrepreneur’s overall portfolio (including both his private equity in the firm and his liquid financial wealth) less risky. This overall portfolio composition effect dominates the high leverage effect within the firm. The more risk-averse the entrepreneur, the stronger the need to reduce his firm risk exposure, therefore the higher the leverage.

Second, on capital budgeting, we extend the standard law-of-one-price-based capital budgeting (net present value (NPV) and adjusted present value (APV) analysis) to account for non-diversifiable risk and incomplete markets. Due to market incompleteness, the entrepreneur will demand an idiosyncratic risk premium when valuing his business. We derive an analytical formula for this idiosyncratic risk premium, the key determinants of which are risk aversion, idiosyncratic volatility, and the sensitivity of entrepreneurial value of equity with respect to cash flow. Quantitatively, we show that ignoring the idiosyncratic risk premium can lead to substantial upward bias in valuation of the firm. One consequence of this bias is that the conventionally used leverage, which does not account for idiosyncratic risk premium, substantially underestimates the leverage of entrepreneurial firms. Based on the survey evidence in Graham and Harvey (2001), smaller firms are less likely to use CAPM than larger ones. One potential explanation is that for the standard arguments (e.g. the life-cycle of firms), smaller firms are also more likely to be run by non-diversified owners/managers who may demand an idiosyncratic risk premium as our theory suggests.
Third, on option valuation, our model extends the Black-Scholes-Merton option pricing methodology to account for the impact of idiosyncratic risk under incomplete markets on (non-tradable) option valuation. The standard dynamic replicating portfolio argument no longer applies, and options can only be valued using utility-based certainty equivalent methodology as we do here. Idiosyncratic volatility now has two opposing effects for option valuation. In addition to the standard positive convexity effect as in Black-Merton-Scholes, the entrepreneur’s precautionary saving motive under incomplete markets implies a negative relation between option value and idiosyncratic volatility, *ceteris paribus*.

The non-diversifiable risk and concentrated wealth in the business make the entrepreneur value his equity less than do diversified investors. Thus, compared to a firm owned by well-diversified investors, the entrepreneur defaults earlier on the firm’s debt, as well as cashes out earlier on his business. Merton (1974) makes the observation that equity is a call option on firm assets, and hence is convex in the firm’s cash flows (under complete markets). Unlike Merton (1974), in our model, inside equity, while also a call option on the entrepreneurial firm’s asset, is not necessarily globally convex in the underlying cash flows. When the entrepreneur’s risk aversion and/or idiosyncratic volatility are sufficiently high, the entrepreneur’s precautionary saving demand can make his private value of equity concave in cash flows.

Fourth, we turn to the agency aspect of our model. Consider the potential risk shifting incentives on investment (as in Jensen and Meckling (1976)). Unlike for the public firm where equityholders have risk seeking incentives (due to the standard option convexity argument of Merton (1974)), the entrepreneur may prefer to invest in a low idiosyncratic volatility project due to his precautionary motive, provided that the firm is not in deep financial distress. This result holds even for very low risk aversion. Our model thus provides a potential explanation for the lack of empirical and survey evidence on asset substitution and risk-shifting incentives (Graham and Harvey (2001)).

Our model generates a rich set of empirical predictions. Consider two otherwise identical firms, one public and one private. First, the private firm will have higher leverage due to diversification arguments. Second, while the standard tradeoff model (e.g. Leland (1994)) predicts leverage decreases with volatility for the public firm, leverage for the private firm might increase with idiosyncratic

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6The usual heterogeneity and endogeneity argument/critique applies. Here, the ownership structure is obviously endogenous.
volatility because of diversification benefits for entrepreneurs. Third, while the complete-market option pricing analysis suggests that higher idiosyncratic volatility defers the exercise of real options in public firms, our model predicts that more idiosyncratic risk makes the private firm have higher default thresholds and lower cash-out thresholds, hence implying a shorter duration to be private. Finally, our model predicts the private firm has more default risk and thus a higher credit spread.

In terms of methodology, our paper also builds on and contributes to the real options literature and the incomplete-markets consumption smoothing/precautionary saving literature (see Deaton (1992) and Attanasio (1999) for surveys). Unlike Miao and Wang (2007), which analyze a real options model under incomplete markets and all equity financing, we integrate incomplete markets real options model with dynamic corporate finance. For analytical tractability reasons, we adopt the CARA utility specification as in Merton (1971), Caballero (1991), Kimball and Mankiw (1989), and Wang (2006). Our model contributes to this literature by extending the CARA-utility-based precautionary saving problem to allow the entrepreneur to reduce his idiosyncratic risk exposure via financing and exit (i.e. cash-out and default) strategies.

2 Model setup

Investment opportunities. An infinitely-lived risk-averse agent has a take-it-or-leave-it project at time 0, which requires a one-time investment $I$. The project generates a stochastic revenue process \{y_t : t \geq 0\} that follows a geometric Brownian motion (GBM):

$$dy_t = \mu y_t dt + \omega y_t dB_t + \epsilon y_t dZ_t, \quad y_0 \text{ given},$$

(1)

where $\mu$ is the expected growth rate of the revenue, $B_t$ and $Z_t$ are independent standard Brownian motions, which are the sources of market (systematic) and idiosyncratic risks of the private business, respectively. The parameters $\omega$ and $\epsilon$ are the systematic and idiosyncratic volatility parameters of

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7Pastor and Veronesi (2005) show that a real option model of IPO decision in an economy with time-varying market conditions does a good job in explaining the IPO waves in the data. Unlike their model, which assumes that the investor faces no idiosyncratic risk before the IPO, our model focuses on the role of idiosyncratic volatility on exits.

8See Brennan and Schwartz (1985) and McDonald and Siegel (1986) for seminal contributions on real options. See Abel and Eberly (1994) for a unified analysis of investment under uncertainty. See Dixit and Pindyck (1994) for a textbook treatment on real options approach to investment.
the revenue growth. The total volatility of revenue growth is

\[ \sigma = \sqrt{\omega^2 + \epsilon^2}. \]  

(2)

As we will show, these different volatility parameters \( \omega, \epsilon, \) and \( \sigma \) have different effects on the entrepreneur’s decision making.

In general, running the project may incur operating costs. In the baseline model, we assume this cost to be zero. In the appendix, we extend our model to account for the flow operating cost, which is assumed to equal to a constant \( z \). The operating cost generates operating leverage and hence positive abandonment option value. Unlike standard abandonment option value, our model incorporates the effect of non-diversifiable risk as we show in the appendix.

In addition, the agent has access to standard financial investment opportunities as in Merton (1971).\(^9\) The agent allocates his liquid financial wealth between a riskfree asset which pays a constant rate of interest \( r \) and a diversified market portfolio (the risky asset) with returns \( R_t \) satisfying:

\[ dR_t = \mu_p dt + \sigma_p dB_t, \]  

(3)

where \( \mu_p \) and \( \sigma_p \) are the expected return and volatility of the risky asset, respectively, and \( B_t \) is the standard Brownian motion introduced earlier. Let

\[ \eta = \frac{\mu_p - r}{\sigma_p} \]  

(4)

denote the after-tax Sharpe ratio of the market portfolio, and let \( \{x_t : t \geq 0\} \) denote the entrepreneur’s liquid (financial) wealth process. The entrepreneur invests the amount \( \phi_t \) in the market portfolio (the risky asset) and the remaining amount \( x_t - \phi_t \) in the riskfree asset.

**Entrepreneurial firm.** If the entrepreneur decides to start the project, he runs it by setting up a limited-liability entity, such as a limited liability company (LLC) or an S corporation. The LLC or S corporation allows the entrepreneur to face single-layer taxation for his business income and

\(^9\)It is straightforward to consider entering the labor market as an alternative to running entrepreneurial business, which provides an endogenous opportunity cost of taking on the entrepreneurial project. Such an extension does not change key economics of our paper in any significant way.
makes the debt non-recourse. We may extend the model to allow for personal guarantee of debt to varying degrees. This feature effectively makes debt recourse to varying degrees. The entrepreneur finances the initial one-time lump-sum cost $I$ via his own funds (internal financing) and external financing. In the benchmark case, we assume that the only source of external financing is debt. See Petersen and Rajan (2002), Heaton and Lucas (2004), and Brav (2009) for evidence that debt is the primary source of financing for most entrepreneurial firms.

One interpretation of the external debt is bank loans. The entrepreneur uses the firm’s assets as collateral to borrow, so that the debt is secured.

We assume that debt is issued at par and is interest-only (consol) for tractability reasons as in Leland (1994) and Duffie and Lando (2001). Let $b$ denote the coupon payment of debt and $F_0$ denote the par value of debt. Debt is priced competitively in that the lender breaks even on the risk-adjusted basis. We further assume that debt is only issued at time 0 and remains unchanged until the entrepreneur exits. Allowing for dynamic capital structure before exit will not change the key economic tradeoff that we focus on: the impact of idiosyncratic risk on entrepreneurial financing decisions.

After debt is in place, at any time $t > 0$, the entrepreneur has three choices: (1) continuing his business; (2) defaulting on the outstanding debt, which leads to the liquidation of his firm; (3) cashing out by selling the firm to a diversified buyer.

While running the business, the entrepreneur receives income from the firm in the form of cash payments (operating profit net of coupon payments). Negative cash payments are interpreted as cash injections by the entrepreneur into the firm. Notice that trading riskless bonds and the diversified market portfolio alone does not help the entrepreneur diversify the idiosyncratic business risk. He can sell the firm and cash out, which requires a fixed transaction cost $K$. The default timing $T_d$ and cash-out timing $T_u$ are not contractible at time 0. Instead, the entrepreneur chooses the default/cash-out policy to maximize his own utility after he chooses the time-0 debt level. Thus, there is an inevitable conflict of interest between financiers and the entrepreneur. The choices of default and cash-out resemble American-style put and call options on the underlying non-tradeable entrepreneurial firm. Since markets are incomplete for the entrepreneur, we cannot price the

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10 In Section 7, we introduce external equity as an additional source of financing.
entrepreneur’s options using the standard dynamic replication argument (Black-Scholes-Merton).

At bankruptcy, the outside lender takes control and liquidates/sells the firm. Bankruptcy *ex post* is costly as in standard tradeoff models of capital structure. We assume that the liquidation/sale value of the firm is equal to a fraction $\alpha$ of the value of an all equity (unlevered) public firm, $A(y)$. The remaining fraction $(1 - \alpha)$ is lost due to bankruptcy costs. We also assume that absolute priority is enforced, and abstract away from any *ex post* renegotiation between the lender and the entrepreneur.

Before the entrepreneur can sell the firm, he needs to retire the firm’s debt obligation at par $F_0$. We make the standard assumption that the buyer is well diversified. He will optimally relever the firm as in the complete-markets model of Leland (1994). The value of the firm after sale is the value of an optimally levered public firm, $V^*(y)$.

After the entrepreneur exits from his business (through default or cash-out), he “retires” and lives on his financial income. He then faces a standard complete-markets consumption and portfolio choice problem.

**Taxes**  We consider a simple tax environment. The entrepreneurial firm pays taxes on his business profits at rate $\tau_e$. When $\tau_e > 0$, issuing debt has the benefit of shielding part of the entrepreneur’s business profits from taxes. For a public firm, the effective marginal tax rate is $\tau_m$. Unlike the entrepreneurial firm, the public firm is subject to double taxation (at the corporate and individual levels), and $\tau_m$ captures the net tax rate (following Miller (1977)). Finally, $\tau_g$ denotes the tax rate on the capital gains upon sale. Naturally, higher capital gain taxes will delay the timing of cash-out.

**Entrepreneur’s objective**  The entrepreneur derives utility from consumption $\{c_t : t \geq 0\}$ according to the following time-additive utility function:

$$\mathbb{E} \left[ \int_0^\infty e^{-\delta t} u(c_t) \, dt \right],$$  \hspace{1cm} (5)

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\[11\] Extending our model to allow for sequential rounds of entrepreneurial activities will complicate our analysis. We leave this extension for future research.
where $\delta > 0$ is the entrepreneur's subjective discount rate and $u(\cdot)$ is an increasing and concave function. The entrepreneur's objective is to maximize his lifetime utility by optimally choosing consumption ($c_t$), financial portfolio ($\phi_t$), and whether to start his business. If he starts his business, he also chooses the financing structure of the firm (coupon $b$), and the subsequent timing decisions of default and cash-out ($T_d, T_u$).

In general, incomplete markets imply that the entrepreneur cannot fully diversify his business risk and hence cannot fully separate his investment from consumption decisions. Indeed, provided that $u'(c)$ is convex, the entrepreneur's precautionary motive will determine his intertemporal consumption smoothing.\(^{12}\)

### 3 Model solution

First, in Section 3.1, we report the complete-markets solution for firm value and financing decisions when the firm is owned by diversified investors. Then, we analyze the entrepreneur's interdependent consumption/saving, portfolio choice, default, and initial investment and financing decisions. The complete-markets solution of Section 3.1 serves as a natural benchmark for us to analyze the impact of non-diversifiable idiosyncratic risk on entrepreneurial investment, financing and valuation.

#### 3.1 Complete-markets firm valuation and financing policy

Consider a public firm owned by diversified investors. Because equityholders internalize the benefits and costs of debt issuance, they will choose the firm’s debt policy to maximize ex ante firm value by trading off the tax benefits of debt against bankruptcy and agency costs. The results in this case are well-known.\(^{13}\) In Appendix A, we provide the after-tax value of an unlevered public firm $A(y)$ in equation (A.19), and the after-tax value of a public levered firm $V^*(y)$ in equation (A.21).

Next, we turn to analyze the entrepreneur’s decision problem before he exits from his business.

\(^{12}\) Leland (1968) is among the earliest studies on precautionary saving models. Kimball (1990) links the degree of precautionary saving to the convexity of the marginal utility function $u'(c)$.

\(^{13}\) For example, see Leland (1994), Goldstein, Ju, and Leland (2001), and Miao (2005).
3.2 Entrepreneur’s problem

The significant lack of diversification invalidates the standard finance textbook valuation analysis for firms owned by diversified investors. As a result, the standard two-step complete-markets (Arrow-Debreu) analysis (i.e., first value maximization and then optimal consumption allocation) no longer applies. This non-separability between value maximization and consumption smoothing has important implications for real economic activities (e.g. investment and capital budgeting) and the valuation of claims that an entrepreneur issues to finance his investment activities.

We solve the entrepreneur’s problem by backward induction. First, we summarize the entrepreneur’s consumption/saving and portfolio choice problem after he retires from his business via either cashing out or defaulting on debt. This “retirement-stage” optimization problem is the same as in Merton (1971), a dynamic complete-markets consumption/portfolio choice problem. Second, we solve the entrepreneur’s joint consumption/saving, portfolio choice, and default decisions when the entrepreneur runs his private business. Third, we determine the entrepreneur’s exit decisions (his cash-out and default boundaries) by comparing his value functions just before and after retirement. Finally, we solve the entrepreneur’s initial (time-0) investment and financing decisions taking his future decisions into account.

Conceptually, our model setup applies to any utility function $u(c)$ under technical regularity conditions. For analytical tractability, we adopt the CARA utility throughout the remainder of the paper. That is, let $u(c) = -e^{-\gamma c}/\gamma$, where $\gamma > 0$ is coefficient of absolute risk aversion, which also measures precautionary motive. We emphasize that the main results and insights of our paper (the effect of non-diversifiable idiosyncratic shocks on investment timing) do not rely on the choice of this utility function. As we show below, the driving force of our results is the precautionary savings effect, which is captured by utility functions with convex marginal utility such as CARA. While CARA utility does not capture wealth effects, it helps reduce the dimension of our double-barrier free-boundary problem, which makes the problem much more tractable compared

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14Cox and Huang (1989) apply this insight to separate intertemporal portfolio choices from consumption in continuous-time diffusion settings.

to constant relative risk aversion (CRRA) utility.

**Consumption/saving and portfolio choice after exit.** After exiting from his business (via either default or cash-out), the entrepreneur no longer has any business income, and lives on his financial wealth. The entrepreneur’s optimization problem becomes the standard complete-market consumption and portfolio choice problem (e.g. Merton (1971)). We summarize the results in Appendix B.

**Entrepreneur’s decision making while running the firm.** We summarize the solution for consumption/saving, portfolio choice, default trigger \(y_d\), and cash-out trigger \(y_u\) in the following theorem.

**Theorem 1** The entrepreneur exits from his business when the revenue process \(\{y_t: t \geq 0\}\) reaches either the default threshold \(y_d\) or the cash-out threshold \(y_u\), whichever comes first. Prior to exit, for given liquid wealth \(x\) and revenue \(y\), he chooses his consumption and portfolio rules as follows:

\[
\bar{c}(x, y) = r\left(x + G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\delta - \gamma r^2}{\gamma r^2}\right),
\]

\[
\bar{\phi}(x, y) = \frac{\eta}{\gamma r \sigma_p} - \frac{\omega}{\sigma_p} y G'(y),
\]

where \(G(\cdot)\) and \(y_d\) solve the free boundary problem given by the differential equation:

\[
r G(y) = (1 - \tau_e) (y - b) + (\mu - \omega \eta)y G'(y) + \frac{\sigma^2 y^2}{2} G''(y) - \frac{\gamma r^2 y^2}{2} G'(y)^2,
\]

subject to the following (free) boundary conditions:

\[
G(y_d) = 0 \quad \text{(9a)}
\]

\[
G'(y_d) = 0 \quad \text{(9b)}
\]

\[
G(y_u) = V^*(y_u) - F_0 - K - \tau_g (V^*(y_u) - K - I) \quad \text{(9c)}
\]

\[
G'(y_u) = (1 - \tau_g) V'^*(y_u) \quad \text{(9d)}
\]

where complete-markets firm value \(V^*(y)\) is defined in \([A.21]\), and the value of external debt \(F_0 = \)
Equation (6) states that consumption is equal to the annuity value of the sum of financial wealth \( x \), certainty equivalent wealth \( G(y) \), and two constant terms capturing the effects of the expected excess returns and the wedge \( \delta - r \) on consumption. The key is to note that \( G(y) \) is the risk-adjusted subjective valuation of the entrepreneur’s business project. Equation (7) gives the entrepreneur’s portfolio holding, where the first term is the standard mean-variance term as in Merton (1971), and the second term gives the entrepreneur’s hedging demand as he uses the market portfolio to dynamically hedge the entrepreneurial business risk.

The differential equation (8) provides a valuation equation for the certainty equivalent wealth \( G(y) \) from the entrepreneur’s perspective. In the standard CAPM model, only systematic risk demands a risk premium under the complete-markets setting. Since the systematic volatility of revenue growth is \( \omega \), the risk-adjusted expected growth rate of revenue in the CAPM model is

\[
\nu = \mu - \omega \eta. \tag{10}
\]

If we drop the last nonlinear term in (8), the differential equation becomes the standard pricing equation: setting the instantaneous expected return of an asset under the risk-neutral measure (RIHS) equal to the riskfree rate (LHS). The last term in (8) captures the additional discount due to the non-diversifiable idiosyncratic risk. Intuitively, the higher the risk aversion parameter \( \gamma \) or the idiosyncratic volatility \( \epsilon_y \), the larger the discount on \( G(y) \) due to idiosyncratic risk. The next section provides more detailed analysis on the impact of idiosyncratic risk on \( G(y) \).

Equation (9a) comes from the value-matching condition for the entrepreneur’s default decision. It states that the private value of equity \( G(y) \) upon default is equal to zero. Equation (9b), often referred to as the smooth-pasting condition, can be interpreted as the optimality condition for the entrepreneur in choosing default.

Now we turn to the cash-out boundary. Because the entrepreneur pays the fixed cost \( K \) and triggers capital gains when cashing out, he naturally has incentive to wait before cashing out. However, waiting also reduces his diversification benefits \textit{ceteris paribus}. The entrepreneur optimally trades off tax implications, diversification benefits, and transaction costs when choosing the timing...
of cashing out. The value-matching condition at the cash-out boundary states that the private value of equity upon the firm’s cashing out is equal to the after-tax value of the public firm value after the entrepreneur pays the fixed costs $K$, retires outstanding debt at par $F_0$, and pays capital gains taxes. The smooth-pasting condition ensures that the entrepreneur optimally chooses his cash-out decision.

**Initial financing and investment decisions.** Next, we complete the model solution by endogenizing the entrepreneur’s initial investment and financing decision. The entrepreneurial firm has two financial claimants: inside equity (entrepreneur) and outside creditors. The entrepreneur values his ownership at a certainty equivalent value $G(y)$. Diversified lenders price debt in competitive capital markets at $F(y)$, which does not contain the idiosyncratic risk premium because outside investors are fully diversified. Thus, the total private value of the entrepreneurial firm is

$$S(y) = G(y) + F(y).$$

We may interpret $S(y)$ as the total value that an investor needs to pay in order to acquire the entrepreneurial firm by buying out the entrepreneur and the debt investors.

At time 0, the entrepreneur thus chooses debt coupon $b$ to maximize the private value of the firm:

$$b^* = \arg\max_b S(y_0; b).$$

Intuitively, the entrepreneur internalizes the benefits and costs of debt financing, and markets competitively price the firm’s debt. In Appendix B, we show that indeed arises from the entrepreneur’s utility maximization problem stated in (B.18). Note the conflicts of interest between the entrepreneur and external financiers. After debt is in place, the entrepreneur will no longer maximize the total value of the firm $S(y)$, but his private value of equity $G(y)$. Theorem 1 has already captured the conflict of interest between the entrepreneur and outside creditors.

The last step is to characterize whether the agent wants to undertake the project. He makes the investment and starts up the firm at time 0 if his life-time utility with the project is higher than that without the project. This is equivalent to the condition that $S(y_0) > I$. 

14
We may interpret our model’s implication on capital structure as a generalized tradeoff model of capital structure for the entrepreneurial firm, where the entrepreneur trades off the benefits of outside debt financing (diversification and potential tax implications) against the costs of debt financing (bankruptcy and agency conflicts between the entrepreneur and outside lenders). The natural measure of leverage from the entrepreneur’s point of view is the ratio between the public value of debt \( F(y) \) and the private value of firm \( S(y) \),

\[
L(y) = \frac{F(y)}{S(y)}.
\]  

We label \( L(y) \) as private leverage to reflect the impact of idiosyncratic risk on the leverage choice. Note that the entrepreneur’s preferences (e.g. risk aversion) influence the firm’s capital structure. The standard argument that shareholders can diversify for themselves and hence diversification plays no role in the capital structure decisions of public firms is no longer valid for entrepreneurial firms.

So far, we have focused on the parameter regions where the entrepreneur first establishes his firm as a private business and finances its operation via an optimal mix of outside debt and inside equity. We now point out two special cases. First, when the cost of cashing out is sufficiently small, it is optimal for the entrepreneur to sell the firm immediately \((y_u = y_0)\). The other special case is when asset recovery rate is sufficiently high, or the entrepreneur is sufficiently risk averse, so that he raises as much debt as possible and defaults immediately \((y_d = y_0)\). Both cases lead to immediate exit. In our analysis below, we consider parameter values that rule out these cases.

4 Risky debt, endogenous default, and diversification

We now investigate a special case of the model in Theorem 1 which highlights the diversification benefits of risky debt. For this purpose, we shut down the cash-out option (by setting the cash-out cost \( K \) to infinity, making the cash-out option worthless).

We use the following (annualized) baseline parameter values: riskfree interest rate \( r = 3\% \), expected growth rate of revenue \( \mu = 4\% \), systematic volatility of growth rate \( \omega = 10\% \), idiosyncratic volatility \( \varepsilon = 20\% \), market price of risk \( \eta = 0.4 \), and asset recovery rate \( \alpha = 0.6 \). We set the
Figure 1: **Private value of equity \( G(y) \): debt financing only.** The top and bottom panels plot \( G(y) \) and its first derivative \( G'(y) \) for \( \tau_e = 0 \) and \( \tau_e = \tau_m \), respectively. We plot the results for two levels of risk aversion (\( \gamma = 1, 2 \)) and the benchmark complete-market solution (\( \gamma \to 0 \)).

In our baseline parametrization, we set \( \tau_e = 0 \), which reflects the fact that the entrepreneur can avoid taxes on his business income completely by deducting various expenses. Shutting down the tax benefits also allows us to highlight the diversification benefits of debt. Later, we consider the case where \( \tau_e = \tau_m \), which can be directly compared with the complete-markets model. We set the entrepreneur’s rate of time preference \( \delta = 3\% \), and consider three values of the risk aversion parameter \( \gamma \in \{0, 1, 2\} \). Finally, we set the initial level of revenue \( y_0 = 1 \).

---

16 We may interpret \( \tau_m \) as the effective Miller tax rate which integrates the corporate income tax, individual’s equity and interest income tax. Using the Miller’s formula for the effective tax rate, and setting the interest income tax at 0.30, corporate income tax at 0.31, and the individual’s long-term equity (distribution) tax at 0.10, we obtain an effective tax rate of 11.29%. 

Private value of equity $G(y)$ and default threshold. Figure 1 plots private value of equity $G(y)$ and its derivative $G'(y)$ as functions of $y$. The top and the bottom panels plot the results for $\tau_e = 0$ and $\tau_e = \tau_m$, respectively. When $\tau_e = 0$, the entrepreneur with very low risk aversion ($\gamma \to 0$) issues no debt, because there are neither tax benefits ($\tau_e = 0$) nor diversification benefits ($\gamma \to 0$). Equity value is equal to the present discounted value of future cash flows (the straight dash line shown in the top-left panel). A risk-averse entrepreneur has incentive to issue debt in order to diversify idiosyncratic risks. The entrepreneur defaults when $y$ falls to $y_d$, the point where $G(y_d) = G'(y_d) = 0$. When $\tau_e = \tau_m$, the entrepreneurial firm issues debt to take advantage of tax benefits in addition to diversification benefits. The bottom two panels of Figure 1 plot this case.

The derivative $G'(y)$ measures the sensitivity of private value of equity $G(y)$ with respect to revenue $y$. As expected, private value of equity $G(y)$ increases with revenue $y$, i.e., $G'(y) > 0$. Analogous to Black-Scholes-Merton’s observation that firm equity is a call option on firm assets, the entrepreneur’s private equity $G(y)$ also has a call option feature. For example, in the bottom panels of Figure 1 ($\tau_e = \tau_m$), when $\gamma$ approaches 0 (complete markets case), equity value is convex in revenue $y$, reflecting its call option feature.

Unlike the standard Black-Scholes-Merton paradigm, neither the entrepreneurial equity nor the firm is tradable. When the risk-averse entrepreneur cannot fully diversify his project’s idiosyncratic risks, the global convexity of $G(y)$ no longer holds, as shown in Figure 1 for cases where $\gamma > 0$. The entrepreneur now has precautionary saving demand to partially buffer against the project’s non-diversifiable idiosyncratic shocks. This precautionary saving effect induces concavity in $G(y)$. When revenue $y$ is large, the precautionary saving effect is large due to high idiosyncratic volatility $\epsilon_y$, and the option (convexity) effect is small because the default option is further out of the money. Therefore, the precautionary saving effect dominates the option effect for sufficiently high $y$, making $G(y)$ concave in $y$ for high $y$. The opposite is true for low $y$, where the convexity effect dominates.

The precautionary saving effect also causes a more risk-averse entrepreneur to discount cash flows at a higher rate. For a given level of coupon $b$, the entrepreneur values his inside equity lower (smaller $G(y)$), thus is more willing to default and walk away. Moreover, a more risk-averse entrepreneur also has a stronger incentive to diversify idiosyncratic risks by selling a bigger share
This table reports the results for the setting where the entrepreneur only has access to debt financing and no option to cash out. The parameters are reported in Section 4. The initial revenue is $y_0 = 1$. We report results for two business income tax rates ($\tau_e = 0\%$, $11.29\%$ ($\tau_m$)) and three levels of risk aversion. The case “$\gamma \to 0$” corresponds to the complete-markets (Leland) model.

| $\gamma \to 0$ | 0.00 | 0.00 | 33.33 | 33.33 | 0.0 | 0 | 0.0 |
| $\gamma = 1$ | 0.31 | 8.28 | 14.39 | 22.68 | 36.5 | 72 | 0.4 |
| $\gamma = 2$ | 0.68 | 14.66 | 5.89 | 20.55 | 71.3 | 166 | 12.1 |

| $\gamma \to 0$ | 0.35 | 9.29 | 20.83 | 30.12 | 30.9 | 75 | 0.3 |
| $\gamma = 1$ | 0.68 | 14.85 | 7.02 | 21.86 | 67.9 | 159 | 9.5 |
| $\gamma = 2$ | 0.85 | 16.50 | 3.77 | 20.27 | 81.4 | 213 | 22.3 |

of his firm, which implies a larger coupon $b$, a higher default threshold, and a higher debt value, *ceteris paribus*. The two effects reinforce each other. Figure 1 confirms that both $G(y)$ and the default threshold $y_d$ increase with risk aversion $\gamma$.

**Capital structure for entrepreneurial firms.** First, we consider the special case where risky debt only offers diversification benefits for the entrepreneur and has no tax benefits ($\tau_e = 0$). Then, we incorporate the tax benefits of debt into our analysis.

The top panel in Table 1 provides results for the entrepreneurial firm’s capital structure when $\tau_e = 0$. If the entrepreneur is very close to being risk neutral ($\gamma \to 0$), the model’s prediction is essentially the same as the complete-market benchmark. In this case, the standard tradeoff theory of capital structure implies that the entrepreneurial firm will be entirely financed by equity (since debt provides no benefits). The risk-neutral entrepreneur values the firm at its market value 33.33.

For $\gamma = 1$, the entrepreneur borrows $F_0 = 8.28$ in market value with coupon $b = 0.31$, and values his non-tradable equity $G_0$ at 14.39, giving the private value of the firm $S_0 = 22.68$. The drop in $S_0$ is substantial (from 33.33 to 22.68, or about 32%) when increasing $\gamma$ from zero to one. This drop
in $S_0$ is mainly due to the risk-averse entrepreneur’s discount of his non-tradable equity position for bearing non-diversifiable idiosyncratic business risks. The default risk of debt contributes little to the reduction of $S_0$ (the 10-year cumulative default probability rises from 0 to 0.4% only).

In Section 3, we introduced the natural measure of leverage for entrepreneurial firms: private leverage $L_0$, given by the ratio of public debt value $F_0$ to private value of the firm $S_0$. Private leverage $L_0$ naturally arises from the entrepreneur’s maximization problem and captures the entrepreneur’s tradeoff between private value of equity and public value of debt in choosing debt coupon policy. For $\gamma = 1$, the private leverage ratio is about 36.5%.

With a higher risk aversion level $\gamma = 2$, the entrepreneur borrows more ($F_0 = 14.66$) with a higher coupon ($b = 0.68$). He values his remaining non-tradable equity at $G_0 = 5.89$, and the implied private leverage ratio $L_0 = 71.3\%$ is much higher than 36.5%, the value for $\gamma = 1$. The more risk-averse entrepreneur takes on more leverage, because he has stronger incentive to sell more of the firm to achieve greater diversification benefits. With greater risk aversion, default is more likely (the 10-year cumulative default probability is 12.1%), and the credit spread is higher (166 basis points over the riskfree rate).

Next, we incorporate the effect of tax benefits for the entrepreneur into our generalized tradeoff model of capital structure for entrepreneurial firms. To compare with the complete-markets benchmark, we set $\tau_e = \tau_m = 11.29\%$. Therefore, the only difference between an entrepreneurial firm and a public firm is that the entrepreneur faces non-diversifiable idiosyncratic risks.

The first row of the lower panel of Table 1 gives the results for the complete-markets benchmark. Facing positive corporate tax rates, the public firm wants to issue debt, but is also concerned with bankruptcy costs. The optimal tradeoff for the public firm is to issue debt at the competitive market value $F_0 = 9.29$ with coupon $b = 0.35$. The implied initial leverage is 30.9% and the 10-year cumulative default probability is tiny (0.3%).

Similar to the case with $\tau_e = 0$, an entrepreneur facing non-diversifiable idiosyncratic risks wants to issue more risky debt to diversify these risks. The second panel of Table 1 shows that the entrepreneur with $\gamma = 1$ borrows 14.85 (with the coupon rate $b = 0.68$), higher than the level for the public firm. The private leverage more than doubles to 67.9%. Not surprisingly, the entrepreneur faces a higher default probability and the credit spread of his debt is also higher. With $\gamma = 2$, debt
This table compares a private firm owned by a risk-averse entrepreneur with a public firm. There is no option to cash out. We assume $\tau_e = \tau_m$, while the rest of the parameters are reported in Section 4. All the results are for initial revenue $y_0 = 1$.

<table>
<thead>
<tr>
<th>$\gamma = 2$ ($b = 0.85, y_d = 0.47$)</th>
<th>10-yr default probability (%)</th>
<th>public debt value</th>
<th>firm value</th>
<th>financial leverage (%)</th>
<th>credit spread (bp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Public ($b = 0.85, y_d = 0.47$)</td>
<td>22.3</td>
<td>16.50</td>
<td>3.77</td>
<td>20.27</td>
<td>81.4</td>
</tr>
<tr>
<td>Public ($b = 0.85, y_d = 0.35$)</td>
<td>22.3</td>
<td>16.50</td>
<td>11.10</td>
<td>27.60</td>
<td>59.8</td>
</tr>
<tr>
<td>Public ($b = 0.35, y_d = 0.14$)</td>
<td>9.8</td>
<td>17.71</td>
<td>11.56</td>
<td>29.26</td>
<td>60.5</td>
</tr>
</tbody>
</table>

is issuance increases to 16.50, and private leverage increases to 81.4%.

**Determinants of capital structure decisions.** To further demonstrate the important role of idiosyncratic risks in determining the capital structure of entrepreneurial firms, we now turn to two comparisons.

First, consider an econometrician who has correctly identified the entrepreneurial firm’s debt coupon $b = 0.85$ and default threshold $y_d = 0.47$, but does not realize that the entrepreneur’s subjective valuation $G(y; b, y_d)$ is lower than the corresponding public equity value $E(y; b, y_d)$ due to non-diversifiable idiosyncratic risk. Indeed, he assigns the entrepreneur’s equity with a value at $E_0 = 11.10$ instead of the subjective valuation $G_0 = 3.77$, thus obtaining a leverage ratio of 59.8%, substantially lower than the entrepreneur’s private leverage $L_0 = 81.4%$. The large difference between the private and market leverage ratios highlights the economic significance of taking idiosyncratic risks into account. Simply put, standard corporate finance methodology potentially underestimates the leverage of entrepreneurial firms.

Second, we highlight the impact of the entrepreneur’s endogenous default decision. The public and the entrepreneurial firms have significantly different leverage decisions because both debt issuance and default decisions on debt (given the same level of debt coupon outstanding) are different. To see the quantitative effects of endogenous default decisions on leverage, we hold the coupon rate on outstanding debt fixed. That is, consider a public firm that has the same technol-
ogy/environment parameters as the entrepreneurial firm. Moreover, the two firms have the same debt coupons \( b = 0.85 \).

Facing the same coupon \( b = 0.85 \), the public firm defaults when revenue reaches the default threshold \( y_d = 0.35 \), which is lower than the threshold \( y_d = 0.47 \) for the entrepreneurial firm. Intuitively, facing the same coupon \( b \), the entrepreneurial firm defaults earlier than the public firm because of the entrepreneur’s aversion to non-diversifiable idiosyncratic risk. The implied shorter distance-to-default for the entrepreneurial firm translates into a higher 10-year default probability (22% for the entrepreneurial firm versus 10% for the public firm) and a higher credit spread (213 basis points for the entrepreneurial firm versus 178 basis points for the public firm). Defaulting optimally for the public firm raises its value from \( S_0 = 27.60 \) to \( S_0 = 29.26 \).

The preceding two comparisons help explain the differences in leverage ratios between the entrepreneurial firm and the public firm. First, fixing both the coupon and the default threshold, the entrepreneur’s subjective valuation (due to non-diversifiable risks) has significant impact on the implied leverage ratio. Ignoring subjective valuation substantially underestimates the entrepreneurial firm’s leverage. Second, facing the same coupon, the entrepreneurial firm defaults earlier than the public firm, which reduces the value of debt and lowers the leverage ratio, but the quantitative effect seems small. Third, diversification motives make the entrepreneur issue more debt than the public firm, which further raises the leverage ratio of the entrepreneurial firm. While the numerical results are parameter specific, the analysis provides support for our intuition that the entrepreneur’s need for diversification and subjective valuation discount for bearing non-diversifiable idiosyncratic risks are key determinants of the private leverage for an entrepreneurial firm.

5 Cash-out option as an alternative channel of diversification

We now turn to a richer and more realistic setting where the entrepreneur can diversify idiosyncratic risks through both the default and cash-out options. The entrepreneur avoids the downside risk by defaulting if the firm’s stochastic revenue falls to a sufficiently low level. In addition, when the firm does well enough, the entrepreneur may want to capitalize on the upside by selling the firm to diversified investors.
In addition to the baseline parameter values from Section 4, we set the effective capital gains tax rate from selling the business $\tau_g = 10\%$, reflecting the tax deferral advantage of the tax timing option. We set the initial investment cost for the project $I = 10$, which is $1/3$ of the market value of project cash-flows. We choose the cash-out cost $K = 27$ to generate a 10-year cash-out probability of about $20\%$ (with $\gamma = 2$), consistent with the success rates of venture capital firms (Hall and Woodward (2008)).

**Cash-out option: Crowding out debt.** Figure 2 plots the private value of equity $G(y)$ and its first derivative $G'(y)$ for an entrepreneur with risk aversion $\gamma = 1$ when he has the option to cash out. The function $G(y)$ smoothly touches the horizontal axis on the left and the dash line denoting the value of cashing out on the right. The two tangent points give the default and cash-out thresholds, respectively. For sufficiently low values of revenue $y$, the private value of equity $G(y)$ is increasing and convex because the default option is deep in the money. For sufficiently high values of $y$, $G(y)$ is also increasing and convex because the cash-out option is deep in the money. For revenue $y$ in the intermediate range, neither default nor cash-out option is deep in the money. In this range, the precautionary saving motive may be large enough to induce concavity. As shown in

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17 In Appendix D.1 we investigate the effects of different capital gain taxes.
Table 3: Capital Structure of Entrepreneurial Firms: debt financing and cash-out option

This table reports the results for the setting where the entrepreneur has access to both public debt financing and cash-out option to exit from his project. The parameters are: \( I = 10, \ K = 27, \ \tau_e = \tau_m = 11.29\%, \ \tau_g = 10\%, \) and \( y_0 = 1. \) The rest of the parameters are from the benchmark case in Section 4.

<table>
<thead>
<tr>
<th>( \gamma \rightarrow 0 )</th>
<th>0.35</th>
<th>9.29</th>
<th>20.83</th>
<th>30.12</th>
<th>30.9</th>
<th>0.3</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma = 1 )</td>
<td>0.55</td>
<td>12.45</td>
<td>9.57</td>
<td>22.02</td>
<td>56.5</td>
<td>4.2</td>
<td>12.3</td>
</tr>
<tr>
<td>( \gamma = 2 )</td>
<td>0.66</td>
<td>13.68</td>
<td>6.24</td>
<td>19.92</td>
<td>68.7</td>
<td>10.1</td>
<td>23.3</td>
</tr>
</tbody>
</table>

the right panel of Figure 1, \( G'(y) \) first increases for low values of \( y \), then decreases for intermediate values of \( y \), and finally increases for high values of \( y \).

Table 3 provides the capital structure information of an entrepreneurial firm with both cash-out and default options. For brevity, we only report the results for the case \( \tau_e = \tau_m \). When the market is complete, the firm’s cash-out option is essentially an option to adjust the firm’s capital structure (recall that there is no diversification benefit for public firms). In this case, given our calibrated fixed cost \( K \), the 10-year cash-out probability is essentially zero and hence this option value is close to zero for the public firm. Therefore, we expect that the bulk of the cash-out option value for entrepreneurial firms comes from the diversification benefits, not from the option value of readjusting leverage.

For a risk-averse entrepreneur, the prospect of cashing out lowers the firm’s incentive to issue debt. When \( \gamma = 1 \), debt coupon is \( b = 0.55 \), lower than the level \( b = 0.68 \) for the firm which only has the default option. The private leverage ratio \( L_0 \) at issuance is 56.5\%, with a credit spread at 138 basis points, compared to the private leverage ratio \( L_0 = 67.9\% \) and credit spread of 159 basis points when the firm only has the default option. The 10-year default probability is close to zero, but the 10-year cash-out probability is 12.3\%, which is economically significant (recall that the 10-year cash-out probability for a public firm is zero). For more risk averse entrepreneurs (e.g. \( \gamma = 2 \)), the private leverage ratio is 68.7\%, smaller than 81.4\% for the setting without the cash-out option.
Given the opportunity to sell his business to public investors, the entrepreneur substitutes away from risky debt and relies more on the future potential of cashing out to diversify his idiosyncratic risks.

**Idiosyncratic risk, leverage, and risk premium.** We now turn to the impact of idiosyncratic volatility on leverage and risk premium. Figure 3 shows its effect on leverage. In complete-markets models, an increase in (idiosyncratic) volatility $\epsilon$ raises default risk, hence the market leverage ratio and the coupon rate for the public firm decrease with idiosyncratic volatility. By contrast, risk-averse entrepreneurs take on more debt to diversify their idiosyncratic risks when $\epsilon$ is higher. For $\gamma = 1$, both coupon and leverage become monotonically increasing in $\epsilon$. This result implies that the private leverage ratio for entrepreneurial firms increases with idiosyncratic volatility even for mild risk aversion.

For public firms, risk premium is determined by the firm’s non-diversifiable systematic risk. For entrepreneurial firm, both systematic and idiosyncratic risks matter for the firm’s investment decisions. Without loss of generality, we decompose the entrepreneur’s risk premium into two com-
ponents: the systematic risk premium $\pi_s(y)$ and the idiosyncratic risk premium $\pi_i(y)$. Rearranging the valuation equation (8) gives:

$$\pi_s(y) = \eta \omega \frac{G'(y)}{G(y)} y = \eta \omega \frac{d \ln G(y)}{d \ln y},$$

(14)

$$\pi_i(y) = \frac{\gamma r}{2} \left( \epsilon y G'(y) \right)^2 \frac{G(y)}{G'(y)}.$$  

(15)

The systematic risk premium $\pi_s(y)$ defined in (14) takes the same form as in standard asset pricing models. It is the product of the (market) Sharpe ratio $\eta$, systematic volatility $\omega$, and the elasticity of $G(y)$ with respect to $y$, where the elasticity captures the impact of optionality on the risk premium.  

Unlike $\pi_s(y)$, the idiosyncratic risk premium $\pi_i(y)$ defined in (15) directly depends on risk aversion $\gamma$ and $(\epsilon y G'(y))^2$, the conditional (idiosyncratic) variance of the entrepreneur’s equity $G(y)$. The conditional (idiosyncratic) variance term reflects the fact that the idiosyncratic risk premium $\pi_i(y)$ is determined by the entrepreneur’s precautionary saving demand, which depends on the conditional variance of idiosyncratic risks (Caballero (1991) and Wang (2006)).

We examine the behavior of these risk premia in Figure 4. The entrepreneur’s equity is a levered position in the firm. When the firm approaches default, the systematic component of the risk premium $\pi_s(y)$ behaves similarly to the standard valuation model. That is, the significant leverage effect around the default boundary implies that the systematic risk premium diverges to infinity when $y$ approaches $y_d$. When the firm approaches the cash-out threshold, the cash-out option makes the firm value more sensitive to cash flow shocks, which also tends to raise the systematic risk premium.

The idiosyncratic risk premium $\pi_i(y)$ behaves quite differently. Figure 4 indicates that the idiosyncratic risk premium is small when the firm is close to default, and it increases with $y$ for most values of $y$. The intuition is as follows. The numerator in (15) reflects the agent’s precautionary saving demand, which depends on the conditional idiosyncratic variance of the changes in the certainty equivalent value of equity $G(y)$ and risk aversion $\gamma$. Both the conditional idiosyncratic risk

\[\text{\footnotesize\hspace{1cm}18}\] Despite this standard interpretation for the systematic risk premium, it is worth pointing out that $\pi_s(y)$ also indirectly reflects the non-diversifiable idiosyncratic risks that the entrepreneur bears, and risk aversion $\gamma$ indirectly affects $\pi_s(y)$ through its impact on $G(y)$.  

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Figure 4: **Systematic and idiosyncratic risk premium.** This figure plots the systematic and idiosyncratic risk premium for entrepreneurial firms. The top panels plot the results for two levels of risk aversion ($\gamma = 2, 4$). The bottom panels plot the results for two levels of idiosyncratic volatility ($\epsilon = 0.20, 0.25$). We assume $\gamma = 2$ (when changing $\epsilon$), $\epsilon = 0.2$ (when changing $\gamma$). The remaining parameters are the same as in Table 8.

variance and $G(y)$ increase with $y$. When $y$ is large, the conditional idiosyncratic variance rises fast relative to $G(y)$, generating a large idiosyncratic risk premium.

6 Project choice: asset substitution versus risk sharing

Jensen and Meckling (1976) point out that there is an incentive problem associated with risky debt: After debt is in place, managers have incentive to take on riskier projects to take advantage of the option-type of payoff structure of equity. However, there is little empirical evidence in support of such risk shifting behaviors. One possible explanation is that managerial risk aversion can potentially dominate the risk shifting incentives. Our model provides a natural setting to investigate these two competing effects quantitatively.

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\textsuperscript{19}See Andrade and Kaplan (1998), Graham and Harvey (2001), among others.
Figure 5: **Private equity value as function of idiosyncratic volatility after optimal debt is in place.** This figure plots the private value of equity for different choices of idiosyncratic volatility $\epsilon$ after debt issuance. The coupon is fixed at the optimal value corresponding to given risk aversion. We assume $\epsilon_{\min} = 0.05$, $\epsilon_{\max} = 0.35$. The remaining parameters are the same as in Table 3.

We consider the following project choice problem. Suppose the risk-averse entrepreneur can choose among a continuum of mutually exclusive projects with different idiosyncratic volatilities $\epsilon$ in the interval $[\epsilon_{\min}, \epsilon_{\max}]$ after debt is in place. Let $F_0$ be the market value of existing debt with the coupon payment $b$. The entrepreneur then chooses idiosyncratic volatility $\epsilon^* \in [\epsilon_{\min}, \epsilon_{\max}]$ to maximize his own utility. As shown in Section 3, the entrepreneur effectively chooses $\epsilon^*$ to maximize his private value of equity $G(y_0)$, taking the debt contract $(b, F_0)$ as given. Let this maximized value be $G^+(y_0)$.

In a rational expectations equilibrium, the lender anticipates the entrepreneur’s *ex post* incentive of choosing the level of idiosyncratic volatility $\epsilon^*$ to maximize $G(y_0)$, and prices the initial debt contract accordingly in competitive capital markets. Therefore, the entrepreneur *ex ante* maximizes the private value of the firm, $S(y_0) = G^+(y_0) + F_0$, taking the competitive market debt pricing into account. We solve this joint investment and financing (fixed-point) problem.

Figure 5 illustrates the solution of this optimization problem. We set $\epsilon_{\min} = 0.05$, $\epsilon_{\max} = 0.35$. When $\gamma \to 0$, the entrepreneur chooses the highest idiosyncratic volatility project with $\epsilon_{\max} = 0.35$. The optimal coupon payment is 0.297. In this case, the entrepreneur effectively faces complete
markets. The Jensen and Meckling (1976) argument applies because the market value of equity is convex and the *ex post risk shifting* problem arises. When the entrepreneur is risk averse, he demands a premium for bearing the non-diversifiable idiosyncratic risks, which tends to lower his private value of equity $G(y_0)$. When this effect dominates, the entrepreneur prefers projects with lower idiosyncratic volatility. For example, for $\gamma = 1$, the entrepreneur chooses the project with $\epsilon_{\min} = 0.05$, with the corresponding optimal coupon payment 0.491. Even when the degree of risk aversion is low (e.g. $\gamma = 0.1$, which implies an idiosyncratic risk premium of 2 basis points for $\epsilon = 0.05$, or 20 basis points for $\epsilon = 0.20$), we still find that the risk aversion effect dominates the risk shifting incentive.

From this numerical example, we find that in our model, even with low risk aversion, the precautionary saving incentive tends to dominate the asset substitution incentive in normal times. Note that our argument applies to public firms as well, provided that: (i) managerial compensation is tied to firm performance; (ii) managers are not fully diversified, behave in their own interests, and are entrenched. Thus, the lack of empirical evidence supporting asset substitution may be simply due to the non-diversifiable idiosyncratic risks faced by risk-averse decision makers.

7 External equity

While debt is the primary source of financing for most entrepreneurial (small-business) firms, high-tech startups are often financed by venture capital, which often use external equity in various forms as the primary source of financing. This financing choice particularly makes sense when the liquidation value of firm’s assets is low (e.g. computer software firms), and incentive alignments are more important. Hall and Woodward (2008) provide a quantitative analysis for the lack of diversification of venture-capital-backed entrepreneurial firms. In this section, we extend the baseline model of Section 2 by allowing the entrepreneur to issue external equity at $t = 0$, and study the effect of external equity on the diversification benefits of risky debt.\(^\text{20}\)

If it is costless to issue external equity, a risk-averse entrepreneur will want to sell the entire

\[\text{Initial equity issuance together with cashing out can be viewed as a two-step procedure to unload the entrepreneur’s private holdings. Gradual sales of ownership, e.g. as DeMarzo and Urosevic (2006) consider for large shareholders, is less applicable for private business owners due to the lack of liquidity, which we capture with the fixed cost $K$.}\]
firm to the VC right away. We motivate the costs of external equity through the agency problems of Jensen and Meckling (1976). Intuitively, the more concentrated the entrepreneur’s ownership is, the better incentive alignment (Berle and Means (1932) and Jensen and Meckling (1976)). Let $\psi$ denote the fraction of equity that the entrepreneur retains and hence $1 - \psi$ denote the fraction of external equity. Consider the expected growth rate of revenue $\mu$ in equation (1). We capture the incentive problem of ownership in reduced form by making $\mu$ an increasing and concave function of the entrepreneur’s ownership $\psi$ (i.e., $\mu'(\psi) > 0$ and $\mu''(\psi) < 0$). Intuitively, the concavity relation suggests that the incremental value from incentive alignment becomes lower as ownership concentration rises, *ceteris paribus*.

More specifically, we model the growth rate $\mu$ as a quadratic function of the entrepreneur’s ownership $\psi$, $\mu(\psi) = -0.02\psi^2 + 0.04\psi + 0.03$, with $\psi \in [0, 1]$. This functional form is chosen such that the maximum expected growth rate is 5%, when the entrepreneur owns the entire firm ($\psi = 1$), while the lowest growth rate is 3%, when the entire firm is sold ($\psi = 0$). The parameters for $\mu(\psi)$ are chosen to keep the agency costs of external equity modest so as to highlight the substitution effect of external equity.

After external debt (with coupon $b$) and equity (with share $1 - \psi$ of the firm ownership) are issued at $t = 0$, the entrepreneur’s optimal policies, including consumption/portfolio rule and default/cash-out policies, are summarized in the following theorem.

**Theorem 2** *The entrepreneur exits from his business when the revenue process $\{y_t : t \geq 0\}$ reaches either the default threshold $y_d$ or the cash-out threshold $y_u$, whichever occurs first. When the entrepreneur runs his firm, he chooses his consumption and portfolio rules as follows:*

\[
\begin{align*}
\tau(x, y) &= r \left( x + \psi G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\delta - r}{\gamma r^2} \right), \\
\phi(x, y) &= \frac{\eta}{\gamma r \sigma_p} - \frac{\psi \omega_0}{\sigma_p} G'(y),
\end{align*}
\]

where $(G(\cdot), y_d, y_u)$ solves the free boundary problem given by the differential equation:

\[
rG(y) = (1 - \tau_c) (y - b - w) + \nu y G'(y) + \frac{\sigma^2 y^2}{2} G''(y) - \frac{\psi \gamma \epsilon^2 y^2}{2} G'(y)^2,
\]
subject to the following (free) boundary conditions:

\[
G(y_d) = 0 \quad (19a)
\]

\[
G'(y_d) = 0 \quad (19b)
\]

\[
\psi G(y_u) = \psi V^*(y_u) - F_0 - K - \tau_g (\psi V^*(y_u) - K - (I - (1 - \psi) E_0)) \quad (19c)
\]

\[
\psi G'(y_u) = (1 - \tau_g) \psi V'^*(y_u) \quad (19d)
\]

where complete-markets firm value \(V^*(y)\) is defined in \(\text{A.21}\), the value of external debt \(F_0 = F(y_0)\) is given in \(\text{C.6}\), and the value of external equity \(E_0 = E(y_0)\) is given by \(\text{C.10}\).

Equation (18) shows how the partial ownership \(\psi\) affects the entrepreneur’s private value of equity. A more concentrated inside equity position (higher \(\psi\)) raises the last nonlinear term, which raises the idiosyncratic risk premium that the entrepreneur demands. The ownership \(\psi\) also affects the boundary conditions at cash-out. The value-matching condition (19c) at the cash-out boundary states that, upon cashing out, the entrepreneur’s ownership is worth fraction \(\psi\) of the after-tax value of the public firm value net of (1) the amount required to retire outstanding debt at par \(F_0\), (2) fixed costs \(K\), and (3) capital gains taxes. The smooth-pasting condition (19d) also reflects the effects of partial ownership.

Finally, at time \(t = 0\), the entrepreneur chooses debt coupon \(b\) and initial ownership \(\psi\) to maximize the private value of the firm \(S(y)\), which now has three parts: inside equity (entrepreneur’s ownership), diversified outside equity, and outside debt:

\[
S(y) = \psi G(y) + (1 - \psi) E_0(y) + F(y). \quad (20)
\]

The results are reported in Table [3]. If the entrepreneur is risk-neutral, he will clearly prefer to keep 100% ownership. In this case, all the equity in the firm is privately held, the private leverage is 33.6%, and the 10-year probabilities of default and cash-out are both close to zero. An entrepreneur with \(\gamma = 1\) lowers his ownership to 69%, which reduces the growth rate to 4.8% (only a 0.2% drop). However, the coupon rises from 0.55 to 0.66, and private leverage rises from 33.6% to 48.8%. The increase in demand for debt due to diversification is economically sizeable, especially considering
This table reports the results for the setting where the entrepreneur has access to both public debt and equity financing, and cash-out option to exit from his project. We assume $\tau_e = \tau_m$, $\mu(\psi) = -0.02\psi^2 + 0.04\psi + 0.03$, while the rest of the parameters are reported in Table 3. All the results are for initial revenue $y_0 = 1$.

<table>
<thead>
<tr>
<th>Ownership</th>
<th>Public debt $F_0$</th>
<th>Public equity $(1-\psi)E_0$</th>
<th>Private equity $\psi G_0$</th>
<th>Private firm $S_0$</th>
<th>Private leverage (%) $L_0$</th>
<th>10-yr default prob (%) $p_d(10)$</th>
<th>10-yr cash-out prob (%) $p_u(10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma \to 0$</td>
<td>1.00</td>
<td>15.23</td>
<td>0.00</td>
<td>30.07</td>
<td>45.30</td>
<td>33.6</td>
<td>0.4</td>
</tr>
<tr>
<td>$\gamma = 1$</td>
<td>0.69</td>
<td>16.00</td>
<td>8.50</td>
<td>8.26</td>
<td>32.76</td>
<td>48.8</td>
<td>3.8</td>
</tr>
<tr>
<td>$\gamma = 2$</td>
<td>0.65</td>
<td>15.93</td>
<td>9.50</td>
<td>5.67</td>
<td>31.10</td>
<td>51.2</td>
<td>6.0</td>
</tr>
</tbody>
</table>

Table 4: Capital Structure of Entrepreneurial Firms: external debt/equity and cash-out option

that the increase is partially offset by the reduced tax benefit of debt due to lower expected growth rates. The 10-year default and cash-out probabilities rise to 3.8% and 11.3% respectively. When $\gamma = 2$, the ownership drops to 65%, while private leverage rises further to 51.2%. The 10-year cash-out probability also rises to 15.4%.

These results demonstrate that entrepreneurial firms have sizeable demand for risky debt and cash-out options for diversification purpose, even when the agency costs of external equity are small.

8 Concluding remarks

Entrepreneurial investment opportunities are often illiquid and non-tradable. Entrepreneurs cannot completely diversify away project-specific risks for reasons such as incentives and informational asymmetry. Therefore, the standard law-of-one-price-based valuation/capital structure paradigm in corporate finance cannot be directly applied to entrepreneurial finance. An entrepreneur acts both as a producer making dynamic investment, financing, and exit decisions for his business project, and as a household making consumption/saving and portfolio decisions. The dual roles of the entrepreneur motivate us to develop a dynamic incomplete-markets model of entrepreneurial finance that centers around the non-diversification feature of the entrepreneurial business.
We show that more risk-averse entrepreneurs use higher leverage for greater diversification benefits and default earlier. This prediction holds not only in the baseline setting where the source of external financing is risky debt, but is robust when we introduce additional channels of diversification such as cash-out option and external equity. In addition to compensation for systematic risks, the entrepreneur also demands sizable premium for bearing idiosyncratic risks, which increase with his risk aversion, his equilibrium inside ownership, and the project’s idiosyncratic variance. Ignoring this idiosyncratic risk premium can lead to a large upward bias when computing the private value of entrepreneurial equity. Finally, even for low to moderate levels of risk aversion, the idiosyncratic risk premium significantly weakens the risk shifting incentives (Jensen and Meckling (1976)) for non-diversified managers.

Our paper also makes methodological contributions to financial valuation and decision making. This paper extends the standard complete-markets Black-Scholes-Merton option pricing and real options methodology to settings where investment opportunities are illiquid and not marked-to-market, and decision makers are not diversified. Our framework can also be used to value the stock options of non-diversified executives or to analyze how these executives make capital structure and investment decisions. See Carpenter, Stanton, and Wallace (2008) for a recent study on the optimal exercise policy for an executive stock option and implications for firm costs.

We have taken a standard optimization framework where the entrepreneur’s utility only depends on his consumption. A significant fraction of entrepreneurs view the non-pecuniary benefits of being their own bosses as a large component of rewards. It has also been documented that less risk-averse (see Gentry and Hubbard (2004), De Nardi, Doctor, and Krane (2007)) and more confident/optimistic individuals are more likely to self-select into entrepreneurship. We also do not model the fundamental frictions causing markets to be incomplete and entrepreneurs to be non-diversified. We view endogenous incomplete markets as a complementary perspective, which can have fundamental implications such as promotion of entrepreneurship and contract design. We leave these extensions for future research.
Appendix

A Market valuation and capital structure of a public firm

Well-diversified owners of a public firm face complete markets. Given the Sharpe ratio $\eta$ of the market portfolio and the riskfree rate $r$, there exists a unique stochastic discount factor (SDF) $(\xi_t : t \geq 0)$ satisfying (see Duffie (2001)):

$$d\xi_t = -r\xi_t dt - \eta\xi_t dB_t, \quad \xi_0 = 1.$$  \hfill (A.1)

Using this SDF, we can derive the market value of the unlevered firm, $A(y)$, the market value of equity, $E(y)$, and the market value of debt $D(y)$. The market value of the firm is equal to the sum of equity value and debt value:

$$V(y) = E(y) + D(y).$$  \hfill (A.2)

Under the risk-neutral probability measure $Q$, we can rewrite the dynamics of the revenue as follows:

$$dy_t = \nu y_t dt + \omega y_t dB_t^Q + \epsilon y_t dZ_t,$$  \hfill (A.3)

where $\nu$ is the risk-adjusted drift defined by $\nu \equiv \mu - \omega \eta$, and $B_t^Q$ is a standard Brownian motion under $Q$ satisfying $dB_t^Q = dB_t + \eta dt$.

A.1 Valuation of an unlevered public firm

Throughout the appendix, we derive our results assuming that there is a flow operating cost $z$ for running the project. The operating cost $z$ generates operating leverage and hence the option to abandon the firm has positive value. The results reported in this paper are for the case $z = 0$. Appendix D.2 provides results for the case $z > 0$.

We start with the after-tax unlevered firm value $A(y)$, which satisfies the following differential equation:

$$rA(y) = (1 - \tau_m) (y - z) + \nu y A'(y) + \frac{1}{2} \sigma^2 y^2 A''(y).$$  \hfill (A.4)
This is a second-order ordinary differential equation (ODE). We need two boundary conditions to obtain a solution. One boundary condition describes the behavior of \( A(y) \) when \( y \to \infty \). This condition rules out speculative bubbles. To ensure that \( A(y) \) is finite, we assume \( r > \nu \) throughout the paper. The other boundary condition is related to abandonment. As in the standard option exercise models, the firm is abandoned whenever the cash flow process hits a threshold value \( y_a \) for the first time. At the threshold \( y_a \), the following value-matching condition is satisfied

\[
A(y_a) = 0,
\]

because we normalize the outside value to zero. For the abandonment threshold \( y_a \) to be optimal, the following smooth-pasting condition must also be satisfied:

\[
A'(y_a) = 0.
\]

Solving equation (A.4) and using the no-bubble condition and boundary conditions (A.5)-(A.6), we obtain

\[
A(y) = (1 - \tau_m) \left[ \left( \frac{y}{r - \nu} - \frac{z}{r} \right) - \left( \frac{y_a}{r - \nu} - \frac{z}{r} \right) \left( \frac{y}{y_a} \right)^{\theta_1} \right],
\]

where the abandonment threshold \( y_a \) is given in

\[
y_a = \frac{r - \nu}{r} \frac{\theta_1}{\theta_1 - 1} z,
\]

where

\[
\theta_1 = -\sigma^{-2} \left( \nu - \sigma^2/2 \right) - \sqrt{\sigma^{-4} \left( \nu - \sigma^2/2 \right)^2 + 2r\sigma^{-2}} < 0.
\]

### A.2 Valuation of a levered public firm

First, consider the market value of equity. Let \( y_d \) be the corresponding default threshold. After default, equity is worthless, in that \( E(y) = 0 \) for \( y \leq y_d \). This gives us the value matching condition \( E(y_d) = 0 \). Before default, equity value \( E(y) \) satisfies the following differential equation:

\[
rE(y) = (1 - \tau_m) (y - z - b) + \nu y E'(y) + \frac{1}{2} \sigma^2 y^2 E''(y), \quad y \geq y_d.
\]
When $y \to \infty$, $E(y)$ also satisfies a no-bubble condition. Solving this ODE and using the boundary conditions, we obtain

$$E(y; y_d) = (1 - \tau_m) \left[ \left( \frac{y}{r - \nu} - \frac{z + b}{r} \right) - \left( \frac{y_d}{r - \nu} - \frac{z + b}{r} \right) \left( \frac{y}{y_d} \right)^{\theta_1} \right]. \quad (A.11)$$

Equation (A.11) shows that equity value is equal to the after-tax present value of profit flows minus the present value of the perpetual coupon payments plus an option value to default. The term $(y/y_d)^{\theta_1}$ may be interpreted as the price of an Arrow-Debreu security contingent on the event of default. The optimal default threshold satisfies the smooth-pasting condition,

$$\frac{\partial E(y)}{\partial y} \bigg|_{y = y_d} = 0, \quad (A.12)$$

which gives

$$y_d^* = \frac{r - \nu}{r} \frac{\theta_1}{\theta_1 - 1} (z + b). \quad (A.13)$$

After debt is in place, there is a conflict between equityholders and debtholders. Equityholders choose the default threshold $y_d$ to maximize equity value $E(y; y_d)$.

The market value of debt before default satisfies the following differential equation:

$$rD(y) = b + \nu y D'(y) + \frac{1}{2} \sigma^2 y^2 D''(y), \quad y \geq y_d. \quad (A.14)$$

The value-matching condition is given by:

$$D(y_d) = \alpha A(y_d). \quad (A.15)$$

We also impose a no bubble condition when $y \to \infty$. Solving the valuation equation, we have

$$D(y) = \frac{b}{r} - \left[ \frac{b}{r} - \alpha A(y_d) \right] \left( \frac{y}{y_d} \right)^{\theta_1}, \quad (A.16)$$

For a given coupon rate $b$ and default threshold $y_d$, using equation (A.2), we may write the
market value of the levered firm value \( V(y; y_d) \) as follows:

\[
V(y; y_d) = A(y) + \frac{\tau_m b}{r} \left[ 1 - \left( \frac{y}{y_d} \right)^{\theta_1} \right] - (1 - \alpha) A(y_d) \left( \frac{y}{y_d} \right)^{\theta_1}, \tag{A.17}
\]

Equation (A.17) shows that the levered market value of the firm is equal to the after-tax unlevered firm value plus the present value of tax shields minus bankruptcy costs.

While \( y_d^* \) is chosen to maximize \( E(y) \), coupon \( b \) is chosen to maximize \textit{ex ante} firm value \( V(y) \). Substituting (A.13) into (A.17) and using the following first-order condition:

\[
\frac{\partial V(y_0)}{\partial b} = 0, \tag{A.18}
\]

we obtain the optimal coupon rate \( b^* \) as a function of \( y_0 \). We also verify that the second order condition is satisfied.

Now consider the special case without operating cost \( (z = 0) \). First, from (A.7), the value of an unlevered public firm becomes

\[
A(y) = (1 - \tau_m) \left[ \frac{y}{r - \nu} - \frac{y_a}{r - \nu} \left( \frac{y}{y_a} \right)^{\theta_1} \right], \tag{A.19}
\]

For a levered public firm, we have an explicit expression for the optimal coupon:

\[
b^* = y_0 \frac{r}{r - \nu} \frac{\theta_1 - 1}{\theta_1} \left( 1 - \theta_1 - \frac{(1 - \alpha)(1 - \tau_m) \theta_1}{\tau_m} \right)^{1/\theta_1}. \tag{A.20}
\]

Substituting (A.13) and (A.20) into (A.17), we obtain the following expression for \( V^*(y) \), the firm value when debt coupon is optimally chosen:

\[
V^*(y) = \left[ 1 - \tau_m + \tau_m \left( 1 - \theta_1 - \frac{(1 - \alpha)(1 - \tau_m) \theta_1}{\tau_m} \right)^{1/\theta_1} \right] \frac{y}{r - \nu}. \tag{A.21}
\]

Note that this firm value formula only applies at the moment of debt issuance and will equal to firm value when the entrepreneur cashes out.
B Proof of Theorem 1 and 2

Theorem 1 is a special case of Theorem 2. Thus, we only prove the results in the general case where the entrepreneur has partial ownership \( \psi \) of the firm.

After exit (via default or cashing out), the entrepreneur solves the standard complete-markets consumption/portfolio choice problem (Merton (1971)). His wealth follows

\[
dx_t = (r (x_t - \phi_t) - c_t) \, dt + \phi_t (\mu_p \, dt + \sigma_p dB_t) . \tag{B.1}
\]

The entrepreneur’s value function \( J^e (x) \) is given by the following explicit form:

\[
J^e (x) = - \frac{1}{\gamma r} \exp \left[ -\gamma r \left( x + \frac{\eta^2}{2\gamma r^2} + \frac{\delta - r}{\gamma r^2} \right) \right] . \tag{B.2}
\]

The consumption and portfolio rules\(^{21}\) are given by

\[
\sigma(x) = r \left( x + \frac{\eta^2}{2\gamma r^2} + \frac{\delta - r}{\gamma r^2} \right) , \tag{B.3}
\]

\[
\bar{\phi}(x) = \frac{\eta}{\gamma r \sigma_p} . \tag{B.4}
\]

Before exit, the entrepreneur’s financial wealth evolves as follows:

\[
dx_t = (r (x_t - \phi_t) + \psi (1 - \tau_e) (y - b - z) - c_t) \, dt + \phi_t (\mu_p \, dt + \sigma_p dB_t) , \quad 0 < t < \min (T_d, T_u) . \tag{B.5}
\]

Using the principle of optimality, we claim that the entrepreneur’s value function \( J^s (x, y) \)

\(^{21}\) An undesirable feature of CARA-utility models is that consumption and wealth could potentially turn negative. Cox and Huang (1989) provide analytical formulae for consumption under complete markets for CARA utility with non-negativity constraints. In our incomplete-markets setting, imposing non-negativity constraints substantially complicates the analysis. Intuitively, requiring consumption to be positive increases the entrepreneur’s demand for precautionary saving because he will increase his saving today to avoid hitting the constraints in the future. The induced stronger precautionary saving demand will likely strengthen our results (such as diversification benefits of outside risky debt).
satisfies the following HJB equation:

$$\delta J^s(x,y) = \max_{c,\phi} u(c) + (rx + \phi(\mu_p - r) - c + \psi(1 - \tau_e)(y - b - z)) J^s_{xx}(x,y)$$

$$+ \mu y J^s_y(x,y) + \frac{(\sigma_p \phi)^2}{2} J^s_{xx}(x,y) + \frac{\sigma^2 y^2}{2} J^s_{yy}(x,y) + \phi \sigma_p \omega y J^s_{xy}(x,y).$$  \hspace{1cm} (B.6)

The first-order conditions (FOC) for consumption $c$ and portfolio allocation $\phi$ are as follows:

$$u'(c) = J^s_s(x,y)$$  \hspace{1cm} (B.7)

$$\phi = \frac{-J^s_s(x,y)}{J^s_{xx}(x,y)} \left( \frac{\mu_p - r}{\sigma^2_p} \right) + \frac{-J^s_{xy}(x,y) \omega y}{J^s_{xx}(x,y) \sigma_p}.$$  \hspace{1cm} (B.8)

We conjecture that $J^s(x,y)$ takes the following exponential form:

$$J^s(x,y) = -\frac{1}{\gamma r} \exp \left[ -\gamma r \left( x + \psi G(y) + \frac{\eta^2}{2\gamma r^2} + \frac{\delta - r}{\gamma r^2} \right) \right].$$  \hspace{1cm} (B.9)

As shown in Miao and Wang (2007), $G(y)$ is the entrepreneur’s certainty equivalent wealth per unit of the entrepreneur’s inside equity of the firm. Under this conjectured value function, it is easy to show that the optimal consumption rule and the portfolio rule are given by (16) and (17), respectively. Substituting these expressions back into the HJB equation (B.6) gives the differential equation (18) for $G(y)$.

We now turn to the boundary conditions. First, consider the lower default boundary. Since equity is worthless at default, the entrepreneur’s financial wealth $x$ does not change immediately after default. In addition, the entrepreneur’s value function should remain unchanged at the moment of default. That is, the following value-matching condition holds at the default boundary $y_d(x)$:

$$J^s(x, y_d(x)) = J^e(x).$$  \hspace{1cm} (B.10)

In general, the default boundary depends on the entrepreneur’s wealth level. Because the default boundary is optimally chosen, the following smooth-pasting conditions at $y = y_d(x)$ must
be satisfied:  

\[
\begin{align*}
\frac{\partial J^s(x, y)}{\partial x} \bigg|_{y=y_d(x)} &= \frac{\partial J^e(x)}{\partial x} \bigg|_{y=y_d(x)} \\
\frac{\partial J^s(x, y)}{\partial y} \bigg|_{y=y_d(x)} &= \frac{\partial J^e(x)}{\partial y} \bigg|_{y=y_d(x)}
\end{align*}
\]  

(B.11)

(B.12)

These two conditions equate the marginal value of wealth and the marginal value of revenue before and after default.

At the instant of cashing out, the entrepreneur retires debt at par, pays fixed cost $K$, and sells his firm for $V^*(y)$ given in (A.17). We assume that the shares owned by existing equity-holders are converted 1 : 1 into the shares of the new firm. Then the entrepreneur pays capital gains taxes on the sale. His wealth $x_{T_u}$ immediately after cashing out satisfies

\[
x_{T_u} = x_{T_u} - \psi V^*(y_{T_u}) - F_0 - K - \tau_g(\psi V^*(y_{T_u}) - K - (I - (1 - \psi)E_0)).
\]  

(B.13)

The entrepreneur’s value function at the payout boundary $y_u(x)$ satisfies the following value-matching condition:

\[
J^s(x, y_u(x)) = J^e(x + \psi V^*(y_u(x)) - F_0 - K - \tau_g(\psi V^*(y_u(x)) - K - (I - (1 - \psi)E_0))).
\]  

(B.14)

The entrepreneur’s optimality implies the following smooth-pasting conditions at $y = y_u(x)$:

\[
\begin{align*}
\frac{\partial J^s(x, y)}{\partial x} \bigg|_{y=y_u(x)} &= \frac{\partial J^e(x + \psi V^*(y) - F_0 - K - \tau_g(\psi V^*(y) - K - (I - (1 - \psi)E_0)))}{\partial x} \bigg|_{y=y_u(x)} \\
\frac{\partial J^s(x, y)}{\partial y} \bigg|_{y=y_u(x)} &= \frac{\partial J^e(x + \psi V^*(y) - F_0 - K - \tau_g(\psi V^*(y) - K - (I - (1 - \psi)E_0)))}{\partial y} \bigg|_{y=y_u(x)}
\end{align*}
\]  

(B.15)

(B.16)

Using the conjectured value function (B.9), we show that the default and cash-out boundaries $y_d(x)$ and $y_u(x)$ are independent of wealth. We thus simply use $y_d$ and $y_u$ to denote the default and cash-out thresholds, respectively. Using the value matching and smooth-pasting conditions (B.10)-(B.12) at $y_d$, we obtain (19a) and (19b). Similarly, using the value-matching and smooth-pasting  

\[22\text{See Krylov (1980), Dumas (1991), and Dixit and Pindyck (1994) for details on the smooth-pasting conditions.}\]
conditions (B.14)-(B.16) at $y_u$, we have (19c) and (19d).

Finally, we characterize the entrepreneur’s investment and financing decision at $t = 0$. Let $x$ denote the entrepreneur’s endowment of financial wealth. If the entrepreneur chooses to start his business, his time-0 financial wealth $x_0$ immediately after financing is

$$x_0 = x - (I - F_0 - (1 - \psi)E_0). \quad (B.17)$$

At time zero, the entrepreneur chooses a coupon rate $b$ and equity share $\psi$ to solve the following problem:

$$\max_{b,\psi} J^s (x + F_0 + (1 - \psi)E_0 - I, y_0), \quad (B.18)$$

subject to the requirement that outside debt and equity are competitively priced, i.e., $F_0 = F(y_0)$, and $E_0 = E_0(y_0)$. In Appendix C, we provide an explicit formulae for $F(y)$ and $E_0(y)$.

The entrepreneur will decide to launch the project if his value function from the project (under the optimal capital structure) is higher than the value function without the project,

$$\max_{b} J^s (x + F_0 + (1 - \psi)E_0 - I, y_0) > J^c (x). \quad (B.19)$$

C Market values of the entrepreneurial firm’s outside debt and equity

When the entrepreneur neither defaults nor cashes out, the market value of his debt $F(y)$ satisfies the following ODE:

$$rF(y) = b + \nu y F'(y) + \frac{1}{2} \sigma^2 y^2 F''(y), \quad y_d \leq y \leq y_u. \quad (C.1)$$

At the default trigger $y_d$, debt recovers the fraction $\alpha$ of after-tax unlevered firm value, in that $F(y_d) = \alpha A(y_d)$. At the cash-out trigger $y_u$, debt is retired and recovers its face value, in that
\( F(y_u) = F_0. \) Solving (C.1) subject to these boundary conditions gives

\[
F(y) = \frac{b}{r} + \left( F_0 - \frac{b}{r} \right) \overline{q}(y) + \left[ \alpha A(y_d) - \frac{b}{r} \right] q(y),
\]  
(C.2)

where

\[
\overline{q}(y) = \frac{y^{\theta_1} y_d^{\theta_2} - y^{\theta_2} y_d^{\theta_1}}{y^{\theta_1} y_d^{\theta_2} - y_u^{\theta_1} y_d^{\theta_2}},
\]  
(C.3)

\[
q(y) = \frac{y^{\theta_2} y_d^{\theta_1} - y^{\theta_1} y_u^{\theta_2}}{y^{\theta_1} y_d^{\theta_2} - y_u^{\theta_1} y_d^{\theta_2}}.
\]  
(C.4)

Here, \( \theta_1 \) is given by (A.9) and

\[
\theta_2 = -\sigma^{-2} \left( \nu - \sigma^2/2 \right) + \sqrt{\sigma^{-4} \left( \nu - \sigma^2/2 \right)^2 + 2r\sigma^{-2}} > 1.
\]  
(C.5)

Equation (C.2) admits an intuitive interpretation. It states that debt value is equal to the present value of coupon payments plus the changes in value when default occurs and when cash-out occurs. Note that \( \overline{q}(y_0) \) can be interpreted as the present value of a dollar if cash-out occurs before default, and \( q(y_0) \) can be interpreted as the present value of a dollar if the entrepreneur goes bankrupt before cash-out. Using \( F_0 = F(y_0) \), we have that the initial debt issuance is given by

\[
F_0 = \frac{b}{r} - \left( \frac{b}{r} - \alpha A(y_d) \right) \frac{q(y_0)}{1 - \overline{q}(y_0)}.
\]  
(C.6)

Similarly, for the outside equity claim, we have the following valuation equation:

\[
rE_0(y) = (1 - r_e) (y - z - b) + \nu y F'(y) + \frac{1}{2} \sigma^2 y^2 F''(y), \quad y_d \leq y \leq y_u,
\]  
(C.7)

subject to the following boundary conditions:

\[
E_0(y_u) = V^*(y_u),
\]  
(C.8)

\[
E_0(y_d) = 0.
\]  
(C.9)
Solving the above valuation equation, we have that the value of outside equity $E_0(y)$ is given by

$$E_0(y) = (1 - \tau_e) \left( \frac{y}{r - \nu} - \frac{z + b}{r} \right) + \left[ V^*(y_0) - (1 - \tau_e) \left( \frac{y_u}{r - \nu} - \frac{z + b}{r} \right) \right] \bar{q}(y)$$

$$- (1 - \tau_e) \left( \frac{y_d}{r - \nu} - \frac{z + b}{r} \right) g(y). \tag{C.10}$$

The initial outside equity issuance $E_0$ is then given by $E_0 = E_0(y_0)$.

D Capital gain taxes and operating leverage

First, we analyze the case where the capital gains tax is zero. Then, we extend the baseline model of Section II to allow for operating leverage.

D.1 Effects of capital gain taxes

In the presence of capital gains taxes with $\tau_g = 0.10$, the benefit from cash-out falls. Table 5 shows that the 10-year cash-out probability decreases, and the entrepreneur takes on more debt in order to diversify idiosyncratic risks. However, the quantitative effects are small in our numerical example. We may understand the intuition from the value-matching condition (9c). At the cash-out threshold $y_u$, when $\psi = 1$, the entrepreneur obtains less value $(1 - \tau_g) V^*(y_u)$, but enjoys tax rebate $\tau_g (K + I)$. Thus, these two effects partially offset each other, making the effect of capital gains taxes small. Clearly, if the cash-out value is sufficiently large relative to the cash-out and investment costs, then the effect of the capital gains tax should be large.

D.2 Effects of operating leverage

How does operating leverage affect an entrepreneurial firm’s financial leverage? Intuitively, operating leverage increases financial distress risk, and thus should limit debt financing. Panel 3 of Table 6 confirms this intuition for the complete-markets case (the limiting case with $\gamma \rightarrow 0$). As the operating cost $z$ increases from 0.2 to 0.4, the 10-year default probability rises from 2.2% to 6.2%, and the firm issues less debt. On the other hand, equity value also decreases because operating costs lower the operating profits. As a result, the effect on financial leverage ratio is ambiguous. In
Table 5: Capital Structure of Entrepreneurial Firms: Capital Gain Taxes

This table reports the results for the setting where the entrepreneur has both default and cash-out options to exit from his project. The parameters are: $r = \delta = 0.03$, $\eta = 0.4$, $\mu = 0.04$, $\omega = 0.1$, $\varepsilon = 0.2$, $\alpha = 0.6$, $I = 10$, and $K = 27$. The initial revenue is $y_0 = 1$. We report results for two business income tax rates ($\tau_e = 0, \tau_m(11.29\%)$), two capital gain tax rates ($\tau_g = 0, 10\%$), and two levels of risk aversion ($\gamma = 1, 2$). The case “$\gamma \to 0$” corresponds to the complete-market model, where the “cash-out” option effectively allows the firm to adjust leverage once.

<table>
<thead>
<tr>
<th>coupon</th>
<th>public debt</th>
<th>private equity</th>
<th>private firm</th>
<th>private leverage (%)</th>
<th>credit spread (bp)</th>
<th>10-yr default probability (%)</th>
<th>10-yr cash-out probability (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_e = 0$, $\tau_g = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1$</td>
<td>0.11</td>
<td>3.20</td>
<td>19.95</td>
<td>23.14</td>
<td>13.8</td>
<td>32</td>
<td>0.0</td>
</tr>
<tr>
<td>$\gamma = 2$</td>
<td>0.42</td>
<td>10.11</td>
<td>10.36</td>
<td>20.47</td>
<td>49.4</td>
<td>115</td>
<td>1.9</td>
</tr>
<tr>
<td>$\tau_e = \tau_m$, $\tau_g = 0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma = 1$</td>
<td>0.54</td>
<td>12.29</td>
<td>9.92</td>
<td>22.22</td>
<td>55.3</td>
<td>138</td>
<td>3.9</td>
</tr>
<tr>
<td>$\gamma = 2$</td>
<td>0.66</td>
<td>13.57</td>
<td>6.47</td>
<td>20.04</td>
<td>67.7</td>
<td>186</td>
<td>9.8</td>
</tr>
</tbody>
</table>

Our numerical examples, this ratio increases with operating costs.

Our analysis above shows that risky debt has important diversification benefits for entrepreneurial firms. This effect may dominate the preceding “crowding-out” effect of operating leverage. Table 6 confirms this intuition. As $z$ increases from 0.2 to 0.4, an entrepreneur with $\gamma = 1$ raises debt with increased coupon payments from 0.59 to 0.62. However, the market value of debt decreases because both the 10-year default probability and the cash-out probability increase with $z$. The private equity value also decreases with $z$ and this effect dominates the decrease in debt. Thus, the private leverage ratio rises with operating costs. This result also holds true for a more risk-averse entrepreneur with $\gamma = 2$. Note that the more risk-averse entrepreneur relies more on risky debt to diversify risk. As a result, the 10-year default probability increases substantially from 26.9% to 50.6% for $\gamma = 2$. But the 10-year cash-out probability decreases from 23.7% to 22.3%.
Table 6: The Effects of Operating Leverage: The case of debt financing and cash-out option

This table reports the results for the setting where the entrepreneur has both default and cash-out options to exit from his project. The parameters are: $r = \delta = 0.03$, $\eta = 0.4$, $\mu = 0.04$, $\omega = 0.1$, $\varepsilon = 0.2$, $\alpha = 0.6$, $\tau_e = \tau_m$, $\tau_g = 10\%$, $I = 10$, and $K = 27$. The initial revenue is $y_0 = 1$. We report results for two levels of risk aversion ($\gamma = 1, 2$) alongside the complete-market solution ($\gamma \to 0$).

<table>
<thead>
<tr>
<th>$\gamma \to 0$</th>
<th>$\gamma = 1$</th>
<th>$\gamma = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>coupon</strong></td>
<td><strong>public</strong></td>
<td><strong>private</strong></td>
</tr>
<tr>
<td>$b$</td>
<td>$F_0$</td>
<td>$G_0$</td>
</tr>
<tr>
<td>$z = 0.2$</td>
<td>0.35</td>
<td>8.03</td>
</tr>
<tr>
<td>$z = 0.4$</td>
<td>0.33</td>
<td>6.72</td>
</tr>
<tr>
<td>$z = 0.2$</td>
<td>0.59</td>
<td>10.94</td>
</tr>
<tr>
<td>$z = 0.4$</td>
<td>0.62</td>
<td>9.41</td>
</tr>
<tr>
<td>$z = 0.2$</td>
<td>0.73</td>
<td>11.95</td>
</tr>
<tr>
<td>$z = 0.4$</td>
<td>0.84</td>
<td>10.48</td>
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References


