On the Analysis of Asymmetric First Price Auctions*

Vlad Mares† Jeroen M. Swinkels‡

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Abstract

We provide a new set of tools for studying asymmetric first price auctions, connecting behavior of equilibria to the $\rho$-concavity of the underlying distributions over costs or values, and showing how one can use surplus expressions related to symmetric auctions to bound behavior in asymmetric auctions. We apply these tools to studying procurement auctions in which, as is common in practice, one seller is given an advantage, reflecting, for example, better reliability or quality. We show conditions under which for any given first price handicap auction, there is a second price auction with bonuses that outperforms it, with most of our results involving ex-post dominance.

Abstract

Keywords: Asymmetric Auctions, Request for Proposal, Differentiation, Mechanism Design, First Price Auctions, Second Price Auctions, Procurement, Rho-concavity.

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†Kellogg School of Management, Northwestern University, Evanston, IL
‡Kellogg School of Management, Northwestern University, Evanston, IL
1 Introduction

Asymmetric first price auctions are notoriously difficult to analyze.\(^1\) This paper provides a set of tools that allows a deeper understanding of such auctions.

We study the case of an auction with two bidders. One of our lead applications is to procurement auctions, and so we will think of sellers 1 and 2 having costs \(c_1\) and \(c_2\) distributed independently according to distributions \(F_1\) and \(F_2\) that need not be symmetric. The results carry over exactly to a standard auction setting with one seller and two buyers with asymmetrically distributed values.

In procurement auctions, it is common for the buyer to view the as the reliability of a given bidder or the quality of the product supplied differently. We model this by including a parameter \(\Delta\) reflecting the amount by which the buyer prefers 1. The buyer may respond to this by setting a handicap \(A\), and running a first price handicap auctions with handicap \(A\) \((\text{FPHA}_A)\) in which the winning bidder receives his bid, but 2 needs to outbid (VV underbid VV) 1 by at least \(A\) to win the job.\(^2\) These auctions, which are very common in practice, create asymmetries even in a setting where underlying costs are symmetric. In particular, handicapping 2 by \(A\) is isomorphic to shifting the distribution over 1’s costs \(A\) to the left. Note that \(\text{FPHA}_0\) is the symmetric first price auction. At the other extreme, consider the commonly used request for proposal \((\text{RFP})\) in which the buyer requests sealed bids on a project and then chooses his ex-post favorite. If \(\Delta\) is known by the bidders, this corresponds to \(\text{FPHA}_\Delta\).\(^3\)

We begin by establishing a connection between the equilibria of asymmetric first price auctions and the local \(p\)-concavity of the distributions over values and of each player’s equilibrium interim expected surplus.\(^4\) The local \(p\)-concavity of a positive function \(h\) at \(x\) is the power one needs to raise \(h\) to make it linear at \(x\).


\(^3\)That \(\Delta\) is common knowledge may or may not be reasonable depending on the context. Analysis of RFPs where \(\Delta\) is unknown to the sellers, or equivalently, handicap auctions with a stochastic allocation rule, is challenging.

\(^4\)As far as we know, \(p\)-concavity is first used in an economic context by Caplin and Nalebuff (1991a, 1991b) who use it in social choice and industrial organization settings.
This connection is important: In any auction setting, interim surplus functions are expressed as integrals of cumulatives (standard cumulatives for standard auctions, reverse cumulatives for procurement auctions). For example, the interim surplus of a bidder with cost \( c \) in a symmetric procurement auction where \( F_1 = F_2 = F \) with support \([0, 1]\) is \( S(c) = \int_0^1 \hat{F}(s) \, ds \), where \( \hat{F} \) is the reverse cumulative of \( F \). So, understanding how properties of a function translate into properties of its integral is of prime interest. The most important theorem about \( \rho \)-concavity is a result by Prekopa (1971,1973) and Borell (1975) relating the \( \rho \)-concavity a function \( h \) and the \( \rho \)-concavity of \( \int h \).

We extend the Prekopa-Borell result in a very useful way. Prekopa and Borell derive a minimum on the \( \rho \)-concavity of \( \int h \) as a function of the \( \rho \)-concavity of \( h \). For an important class of functions that arises naturally in an auction theoretic context, we provide a counterpart result showing a maximum on the \( \rho \)-concavity of \( \int h \) as a function of the \( \rho \)-concavity of \( h \). We also show how to use these results to derive useful connections between the monotonicity of the local \( \rho \)-concavities of \( h \) and of \( \int h \).\(^5\)

Connecting up the properties of \( \hat{F} \) and of \( S(c) \) allows us to derive other properties of the equilibrium. For example, for symmetric auctions, we show how to connect up the \( \rho \)-concavity of \( \hat{F} \) to the slope of the bidding functions. So, in particular, we can think about how cost savings by the seller are divided between the seller and the buyer. Similarly, we explore when better costs do or do not lead to more aggressive bidding for any given cost. And, while the bulk of the paper is about two player auctions, we show how one can use the properties of \( \rho \)-concavity to generate a new comparative static on equilibria in a symmetric setting as the number of bidders changes.

For asymmetric auctions, we derive useful bounds on \( S \) by exploiting the connection between \( F \) and \( \int \hat{F} \), and by linking surplus for the asymmetric auction to surplus for a particular symmetric auction. Using these bounds, we are able to derive new results on the slopes and shape of equilibrium bid functions, and the allocations they generate. In particular, we can say a fair bit about the geometry of the function \( \phi_{FP_A}(c_2) \) which, for any given handicap \( A \), divides those cost types \( c_1 \) with which 1 wins from those with which 1 loses.

We think these tools will have broader applicability to auction theory and mechanism design. But, for much of the paper, we focus on a specific

\(^5\)A very similar set of results about \( \rho \)-concavity has independently been discovered by Weyl and Fabinger (2009), who use them to study the question of “pass-through” in oligopoly settings. We discuss this further below.
application: we study the relative merits of the first price handicap auction and a second price auction in which the seller can also choose an amount $A$ by which to favor one seller. In particular, consider a simple second price bonus auction with bonus $A$ (SPBA$_A$), in which the low bidder wins, but if 2 wins, he receives $b_1$ while if 1 wins, he receives $b_2 + A$. The use of bonuses of this form is fairly prevalent: it is common to write, as part of the rules for a second price (or open) auction, who is responsible for various costs associated with the work, such as specialized tooling. As such, one can have the extra costs involved with a new supplier competing against an old born by the buyer, the seller, or some combination. As a practical matter, it is these simple mechanisms that are observed, making a comparison of their merits of substantial importance.\footnote{One reason why the simplicity of a fixed bonus $A$ may be favored in practice (as opposed, say to simply implementing the optimal mechanism) is that complicated rules can be susceptible to interim strategic manipulation by the buyer. Another reason is that it may be either illegal or unpalatable to the bidders to write rules that explicitly favor one bidder as a function of the other costs, but easier to write rules of the form “the buyer will pay transportation costs” implicitly favoring one bidder, but in a coarser manner.}

First price handicap auctions in this setting have an odd feature. Imagine, for simplicity, that costs have common support $[0, 1]$. Then, $\beta_1(0) = \beta_2(0) + A$, where $\beta_i$ is the bid function of $i$.\footnote{In some settings, an open RFP is used, with multiple rounds of offers. An example is the 2003 competition between Boeing and Airbus to sell plane to Iberia Airlines. If $A$ is common knowledge, this is equivalent to a second price auction with bonus $A$ paid to 1, since he needs only match the $c_2$ plus $\Delta$ to be picked.} Thus, when costs are low, all of the allocative effects of the handicap format are undone by equilibrium effects. On the other hand, we will also show that when 2 has costs above $1 - A$, bidder 1 wins for sure. So, in equilibrium, the FPHA creates lots of distortion away from the symmetric case when costs are high, but very little when costs are low.

A second price mechanism, of course, creates a uniform distortion away from the symmetric cost case. One way to rank the two auctions is thus to ask what sort of distortion an optimal mechanism (Myerson (1981), Riley and Samuelson (1981)) would have specified. Here again, $\rho$-concavity is at the heart of the matter. For a large class of cost distributions (ones with a particular monotonicity on the $\rho$-concavity of $F_1$ or $F_2$), the optimal mechanism specifies a distortion that is smaller at higher costs. So, by inducing no distortion at low costs, but a lot of distortion at high costs, the first price handicap auction is getting things backward for such distributions.

\footnote{A bid by 1 below $\beta_2(0) + A$ or by 2 below $\beta_1(0) - A$ can be raised a little and still win for sure, and so is not optimal.}
at least at extreme costs.

The heart of the development is to show conditions under which this translates to the more demanding condition that the distortion created by the first price auction is monotonically increasing in costs. Under those conditions, for any first price mechanism, with handicap $A_{FP}$, there is a second price mechanism, with bonus $A_{SP}$, such that, for each $(c_1, c_2)$, the outcome achieved by the second price mechanism is never worse, and sometimes better, than that of the first price mechanism. Essentially, since the first price auction is creating an increasing distortion, while the optimal mechanism would specify a decreasing one, an appropriately chosen second price mechanism can split the difference, and create an allocation which agrees with the optimal allocation on a strict superset of cost realizations.

Our conditions are most easily satisfied when $\Delta > 0$, but $F_1 = F_2$. But, we also make considerable headway when $F_1$ and $F_2$ differ. The key to a result is that, in a sense to be made precise, $F_1$ can be viewed as a stretch and convexification of $F_2$, and that $\Delta$ is not too small relative to the asymmetry between $F_1$ and $F_2$.

As we will argue in more detail later, we think these results have significant practical importance. While our model is surely not general enough to capture all procurement settings, it seems a fair approximation to many in which first price mechanisms are used. There is serious reason for the users of these mechanisms to reconsider their choice.

Section 2 discusses the literature. Section 3 presents the model. Section 4 derives the main technical results related to $\rho$-concavity. Section 5 examines first price mechanisms and their basic equilibrium properties and provides some quick implications of the connection to $\rho$-concavity. Section 6 looks at the choice between first and second price mechanisms. Section 7 connects our ranking results to a key paper by Maskin and Riley (2000a). Section 8 concludes. As much of the contribution of this paper is technical, illuminating proofs are kept in main text. The balance are in the appendices.

2 Related Literature

A good entry point into the general literature on asymmetric first price auctions is Maskin and Riley (2000a,2000b). Lebrun has done extensive work characterizing equilibria in asymmetric first price auctions, showing (1996,1999) that equilibria exist if the signals are asymmetrically distributed with common compact support, and (1998) that in a two-bidder common-support first price auction the bidder who draws his signal from a better dis-
tribution (in the sense of hazard rate dominance) will bid more aggressively. The equilibrium in these auctions is unique under log-concave distributions (Maskin and Riley (2003), Lebrun (2006)). A key result on existence of monotone strategies in first price auctions is provided by Reny and Zamir (2004).\footnote{For a further summary of the general literature on asymmetric first price auctions, see Krishna (2002) and the references therein.}

Myerson (1981) and Riley and Samuelson (1981) begin a long discussion of implementation with asymmetric cost distributions. McAfee and McMil lan (1988) look at optimal mechanisms with asymmetric cost distributions and argue that one doesn’t want to always buy from the low-cost bidder.

In Che (1993), suppliers have different costs and can provide goods of different qualities. Suppliers submit a bid \((p, q)\) which is evaluated via a quasi-linear scoring rule \(S(p, q)\). Under the \textit{first score rule} the high scorer executes the submitted bid. Under the \textit{second score rule} the high scorer executes a contract equivalent in score and cost to the next highest bid. This transforms a multidimensional problem into a standard asymmetric cost environment. The optimal scoring rule distorts quality downward, and can be implemented by either the first or second score rules. Naegelen (2002) extends Che’s result by allowing for an exogenous preference for one bidder. Branco (1997) adds common value aspects and correlation to costs. Asker and Cantillion (2006) expand the results to multi-dimensional quality. Ganuza and Pechlivanos (2000) consider a setting in which, by design of the object to be procured, the buyer can affect the degree to which one firm or another has an advantage. They find that when the mechanism can be asymmetric, the buyer will choose a design which favors one firm, but then use the mechanism to “recapture” that advantage, while if the buyer must use a symmetric mechanism, he will choose a design that increases homogeneity, and thus competition. In our setting, the advantage of one supplier over another is exogenous, and the question is which of two simple mechanisms will do better.

Manelli and Vincent (1995) also study procurement settings. In their model, low costs are correlated with low quality. Our setting differs from theirs in that we assume that the buyer knows his relative valuation for purchasing from 1 and 2, and so does not update on their cost realization.

Shachat and Swarthout (2003) consider a simple model with a uniform distributions over costs and over qualities. They provide experimental evidence that the optimal English auction with a bonus outperforms a standard sealed bid RFP setting (in a setting where bidders are aware of their own
quality but not that of their opponent).

In an empirical context Marion (2006) estimates the price effect of favoring disadvantaged bidders through proportional bonuses in federal procurement auctions at 3.8%. While this price effect is small the approach ignores any efficiency considerations which are explicit in our model. On the other hand, Marion (2007) points out that bid preferences can have significant negative participation effects on disadvantaged bidders, a problem which we do not address in this paper. Cabral and Greenstein (1990) discuss other empirical implications of favored bidding in federal procurement.

Our results on the relative performance of first and second price auctions have interesting connections to Maskin and Riley (2000a). We discuss this in detail in Section 7.

3 Model

A buyer faces sellers 1 and 2 with costs $c_1$ and $c_2$. Costs are independent, from cumulatives $F_1$ and $F_2$, where $F_i$ has density $f_i$ which is log-concave and continuously differentiable. We assume costs have finite support and normalize the support of $F_2$ without loss of generality to $[0, 1]$. The support of $F_1$ is $[c_1, c_1]$. The reverse cumulative is $\bar{F}_i = 1 - F_i$. We assume that $\frac{\bar{F}_i}{F_i}$ is strictly increasing. This is very mild. The buyer’s utility from purchasing from $i$ is

$$U_B(p, i) = v_i - p,$$

where $p$ is the transaction price. Let $\Delta = v_1 - v_2$ be the amount by which 1 is preferred to 2. We assume the buyer knows $\Delta$ at the time he chooses the rules of the auction. Without loss of generality, we assume $\Delta \geq 0$.

10 “Through various preference programs, the U.S. federal government in 2001 awarded $21.3 billion of procurement contracts to small firms, minority and women owned businesses, companies located in economically disadvantaged areas, and veteran-owned businesses. This represents nearly 10 percent of the $216 billion federal procurement market.”

11 Given the normalization of the support of $F_2$, it can be that $c_1 < 0$. This leads to no problems: shifting both distributions and the buyer’s value by a constant does not change any of our results.

12 By log-concavity, $\frac{\bar{F}}{F}$ is weakly increasing (Lemma 21). If $\frac{\bar{F}}{F} = \gamma > 0$ on $[a, b]$, then $\bar{F}(c) = \bar{F}(a)e^{-\gamma c}$ for $c \in [a, b]$. Hence, we are ruling out that the distribution has a segment of an exponential distribution “patched” into it.

13 For formal auctions, the rules of the auction specify $A$, at which point $\Delta$ is irrelevant to the sellers. As discussed, the connection between an RFP and a FPHA requires $\Delta$ to be common knowledge between the bidders.

14 Rezende (2004) asks the rather interesting question of whether the buyer should want
For general functions $j$ and $k$ on given domain, we write $j \geq k$ if $j(c) \geq k(c)$ everywhere on the domain, and $j > k$ if $j(c) > k(c)$ everywhere. By $(j')'(k(x))$, we mean the derivative of $j$ evaluated at $k(x)$. We say that $j$ is increasing (decreasing) if $j' \geq (\leq) 0$ everywhere. Write $j(c) =_s k(c)$ if $j(c)$ has the same sign as $k(c)$, and $j(c) \geq_1 k(c)$ if $j(c)$ is positive any time $k(c)$ is.

4 Local Concavity

A positive $C^2$ function $h : [0, 1] \rightarrow \mathbb{R}_+$ is $\rho$-concave if

$$ \frac{\left( \frac{h''}{\rho} \right)^n}{0}. $$

Standard concavity is equivalent to 1-concavity, log-concavity is equivalent to 0-concavity. See Prekopa (1971,1973) and Borell (1975)

It is straightforward that $\frac{h''}{\rho}$ is concave at $x$ iff

$$ \rho \leq 1 - W_h(x). $$

where

$$ W_h(x) \equiv \frac{h''}{(h')^2}(x) $$

We can thus define $\rho_h(x) \equiv 1 - W_h(x)$ as the local $\rho$-concavity of $h$ at $x$. By definition, a function $h$ is $\rho$-concave if $\rho_h(x) \geq \rho$ for all $x$.

Define

$$ \bar{H}(c) = \int_c^1 h(s) ds. $$

We will assume that any $h$ we deal with is sufficiently well behaved that $W_h(1) \equiv \lim_{x \to 1} W_h(x)$ and $W_{\bar{H}}(1) \equiv \lim_{x \to 1} W_{\bar{H}}(x)$ are well defined in the extended real line. We will also assume that when $h(1) = 0$, $W_h(1)$ is finite. This is extremely mild. See the first appendix for a discussion and primitives.

Perhaps the most important property of $\rho$-concavity is the following result due to Prekopa and Borell.

**Proposition 1** If $h$ is $\rho$ concave, then $\bar{H}$ is $\frac{\rho}{1 + \rho}$ concave. That is,

$$ \rho_{\bar{H}}(c) \geq \frac{\rho}{1 + \rho}. $$

to know $\Delta$. In a setting where the buyer cannot commit to a mechanism, the answer can easily be no.
We can strengthen this result in a very useful way. Let
\[ h(c) = \max_{s \in [c, 1]} h(s) \quad \text{and} \quad h(c) = \min_{s \in [c, 1]} h(s), \]
and let
\[ W_h(c) = \min_{s \in [c, 1]} W_h(s) \quad \text{and} \quad W_h(c) = \max_{s \in [c, 1]} W_h(s), \]
so that \( W_h(c) \) is weakly increasing, and \( W_h(c) \) is weakly decreasing. Then,

**Theorem 1** Let \( h \) be \( \rho \)-concave with \( \rho > -1 \). Let \( h \) be decreasing on some \([\hat{c}, 1]\) and \( h(1) = 0 \). Then,
\[
\frac{\tilde{h}(c)}{1 + \tilde{h}(c)} \geq \rho H(c) \geq \frac{\check{h}(c)}{1 + \check{h}(c)}
\]
for all \( c \in [\hat{c}, 1] \).

This is stronger than Proposition 1 in the (not very surprising) dimension that only the properties of \( h \) on one side of \( c \) matter, but more importantly in bounding the \( \rho \)-concavity of \( H \) from above. One common use of Theorem 1 is when \( h(c) = F_i(c) \), and so \( H(c) = \int_{c}^{1} F_i(s) \, ds \). To prove Theorem 1, we begin with two lemmas.

**Lemma 1** An equivalent expressions to (1) is
\[
2 - \tilde{W}_h(c) \leq \frac{1}{W_H(c)} \leq 2 - W_h(c).
\]

**Lemma 2** If \( h \) is \( \rho \)-concave and \( h(1) = 0 \) then
\[
2 - \tilde{W}_h(1) = \frac{1}{W_H(1)} = 2 - W_h(1).
\]

**Proof of Theorem 1** We work with (2). Let
\[
\tilde{J}(c) \equiv -\frac{1}{W_H(c)} + 2 - W_h(c),
\]
\[
J(c) \equiv -\frac{1}{W_H(c)} + 2 - W_h(c),
\]
and
\[
\check{J}(c) \equiv -\frac{1}{W_H(c)} + 2 - \check{W}_h(c).
\]
so that $\bar{J}(c) \leq J(c) \leq \bar{J}(c)$. Note that

$$W_H'(c) = W_H(c) \left( \frac{-h(c)}{H(c)} - \frac{h'(c)}{h(c)} \right)$$

$$= W_H(c) \left( \frac{-h'(c)}{h(c)} \right) \left( \frac{-1}{W_H(c)} + 2 - W_H(c) \right)$$

and that

$$W_H(c) \left( \frac{-h'(c)}{h(c)} \right) = \bar{H}(c) \left( \frac{-h'(c)}{h(c)} \right)^2 > 0,$$

and so

$$W_H'(c) = s J(c). \tag{4}$$

Assume that at some $c$, $\frac{1}{W_H(c)} > 2 - W_H(c)$, or, equivalently, $\bar{J}(c) < 0$. Then, $J(c) \leq \bar{J}(c) < 0$, and so

$$W_H'(c) < 0.$$

Since $W_h(c)$ is weakly increasing,

$$J'(c) = \frac{W_H'(c)}{W_H(c)} - W_h'(c) < 0$$

as well. Thus, if $J(c) < 0$, $J'(c) < 0$, and it follows that $J'(s) < 0$ for all $s \in [c, 1]$. But then, $\bar{J}(1) < 0$, contradicting Lemma 2.

Similarly, assume $2 - W_h(1) > \frac{1}{W_H(1)}$, or equivalently $J(c) > 0$. Then, $W_H'(c) > 0$ and so since $W_h'(c) \leq 0$,

$$J'(c) = \frac{W_H'(c)}{W_H(c)} - W_h'(c) > 0.$$

Thus, $J(c) > 0$, again contradicting Lemma 2. \hfill \blacksquare

Note that what has been proved here is $\bar{J}(c) \geq 0$, and $\bar{J}(c) \leq 0$. Together with (4), we thus have the following corollary.

**Corollary 1** Assume $h(1) = 0$. If on some interval $[\bar{c}, 1]$, $h$ is decreasing while $\rho_h$ is decreasing, then $\rho_H$ is decreasing on $[\bar{c}, 1]$. If $h$ is decreasing, while $\rho_h$ is increasing on $[\bar{c}, 1]$, then $\rho_H$ is increasing on $[\bar{c}, 1]$.

This is immediate, since if $W_h$ is increasing then $W_h(c) = W_h(c)$, and so $J(c) = \bar{J}(c) \geq 0$, and similarly if $W_h$ is decreasing. As a useful adjunct to this, we have
Lemma 3 If $h$ is log-concave and increasing at $c$ then $W_H$ is increasing at $c$.

To see this, note that if $h$ at $c$ is increasing and log-concave then $\frac{h'}{h}$ is positive and decreasing, as is $\frac{H'}{H}$. Thus, $-W_H = \frac{h'\ln}{h}$ is decreasing.

Corollary 2 If $h$ is increasing on $[c, 1]$, then $W_H = 1 - \rho_H \leq \frac{1}{2}$. If $f$ is decreasing on $[c, 1]$ then $W_H = 1 - \rho_H \geq \frac{1}{2}$.

To see this, note that where $f_i$ is increasing, $F_i$ is concave, and so $\rho_{F_i}(c) \geq 1$. Similarly, where $f$ is decreasing, $F_i$ is convex, and so $\rho_{F_i}(c) \leq 1$. The claim is then immediate from Theorem 1.

Weyl and Fabinger (2009, Theorem 3) independently obtain a very similar result. They show that $W_{fF}(c)$ can be expressed as an integral of the form $\int_c^1 \frac{q(s)ds}{2 - W_{fF}(s)}$ which allows for an alternative way of deriving the extension of the Prekopa-Borell result.

The main focus of their paper is to study cost pass-through rates in oligopoly settings. The fact that they find $\rho$-concavity (in their case of a demand function) at the heart of their results strengthens our belief in the importance of $\rho$-concavity for studying this sort of problem (note in particular that, along the lines of Bulow and Roberts (1989), there is a strong relationship between a demand curve in a monopoly problem and a type distribution in an auction setting).

Two fascinating further points arise in that paper. First, Weyl and Fabinger observe that the vast majority of commonly used functional forms used in empirical work to estimate demand functions satisfy our monotonicity requirement on $W_{fF}$. Second, it turns out that an expression of the form $W$ shows up pretty early in economic analysis! Cournot uses it in his 1838 study of the oligopoly problem.

5 First Price Mechanisms

In a First Price Handicap Auction with handicap $A$ (FPHA$_A$) bidders 1 and 2 draw costs independently from $F_i$, and submit bids $b_1, b_2$. Bidder 1 wins iff $b_1 < b_2 + A$. The winner receives their bid. Player $i$ is restricted to bid at most $\tilde{c}_i$, so that they cannot receive a payment larger than their highest possible cost.\textsuperscript{15}

\textsuperscript{15}Depending on the relation between $F_1$ and $F_2$, one of these will be irrelevant.
For the purposes of interpreting our results consider two related settings. In a *First Price Bonus Auction with bonus* $A$, 1 and 2 draw costs independently from $F_i$ and submit bids $b_1, b_2$. Bidder 1 wins iff $b_1 < b_2$. If 2 wins, he receives $b_2$. If 1 wins, he receives $b_1 + A$. Player 1 is restricted to bid at most $c_1 - A$. In a *First Price Shifted Cost Auction with shift* $A$, 1 and 2 draw costs independently and submit bids $b_1, b_2$. Bidder 2 draws his cost from $F_2$. Bidder 1 draws his cost from $F_{1,A}$ defined on $[c_1 - A, c_1 - A]$ by

$$F_{1,A}(c_1) = F_1(c_1 + A).$$

Bidder 1 wins iff $b_1 < b_2$. The winner receives their bid. Player 1 is restricted to bid at most $c_1 - A$.

There is an obvious isomorphism between the auctions above, so that an allocation is an equilibrium in one iff its obvious translation is an equilibrium allocation in the other.\footnote{See Mares and Swinkels (2008) for details.}

We consider Bayesian equilibria in which $b_i \geq c_i$,\footnote{Ruling out such weakly dominated strategies rules out uninteresting pathologies.} and assume that equilibria are in pure, continuous, and strictly increasing strategies. Primitives for this can be found in Reny and Zamir (2004) and Jackson and Swinkels (2005) (with each applied to the shifted cost auction with shift $A$).\footnote{See also Lebrun (1996,2006), who assumes costs have identical support. See Mares and Swinkels (2008) for further detail.}

Given $FPHA_A$, Define $\phi_{FP}$ (FP mnemonic for first price) as

$$\beta_1(\phi_{FP}(c_2)) = \beta_2(c_2) + A.$$

Since the equilibrium is strictly increasing, $\phi_{FP}$ is well-defined and increasing. If $c_1 > \phi_{FP}(c_2)$, then 2 wins in equilibrium, while if $c_1 < \phi_{FP}(c_2)$, then 1 wins. Define $\psi_{FP}$ as the inverse of $\phi_{FP}$\footnote{Here and in a number of analogous situations that follow, it can be the case that for some $c_2$, $\beta_1(c_1) < \beta_2(c_2) + A$ for all $c_1$. In this event, define $\phi_{FP}(c_2) = c_1$. Similarly, (although this will not in fact occur in this instance) if it were the case that for some $c_2$, $\beta_1(c_1) > \beta_2(c_2) + A$ for all $c_1$, one would define $\phi_{FP}(c_2) = c_1$.}.

**Proposition 2** Let $\beta_1, \beta_2$ be an equilibrium of $FPHA_A$. Then $\beta_1(\xi_1) = \beta_2(0) + A$, or, equivalently, $\phi_{FP}(0) = \xi_1$.

This follows immediately, noting that since a bid of $\beta_2(0) + A$ by 1 wins with probability one it dominates any lower bid, and similarly for a bid of $\beta_1(\xi_1) - A$ by 2.
5.1 Basic Characterization

We now turn to a more detailed examination of the equilibrium bid and allocation functions. We will assume that $c_1 - A < 1$. The case $c_1 - A \geq 1$ is largely similar.

**Theorem 2** In FPHA, 1’s surplus with cost $c$ is

$$S_1(c) = \int_c^{c_1} \bar{F}_2(\psi(s))ds. \quad (6)$$

Player 2’s surplus with cost $c$ is

$$S_2(c) = \int_c^{c_1-A} \bar{F}_1(\phi(s))ds. \quad (7)$$

Bid functions are

$$\beta_1(c) = c + \frac{\int_c^{c_1} \bar{F}_2(\psi(s))ds}{\bar{F}_2(\psi(c))}$$

and

$$\beta_2(c) = c + \frac{\int_c^{c_1-A} \bar{F}_1(\phi(s))ds}{\bar{F}_1(\phi(c))}. \quad (9)$$

If $f_1$ and $f_2$ are $C^k$, then $\beta_1, \beta_2, \text{ and } \phi$ are $C^{k+1}$. On their domains

$$\beta_1(c) = \frac{1}{\phi'(\psi(c))} S_1(c) \frac{f_2(\psi(c))}{\bar{F}_2(\psi(c))} > 0, \quad (10)$$

$$\beta_2(c) = \phi'(c) S_2(c) \frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} > 0, \quad (11)$$

and

$$\phi'(c) = \frac{S_1(\phi(c))}{S_2(c)} \frac{f_2(c)}{f_1(\phi(c))} > 0. \quad (12)$$

Equations (6) and (7) follow from the envelope theorem, noting that $\bar{F}_2(\psi(s))$ is the probability that 1 wins with value $s$ and $\bar{F}_1(\phi(s))$ is the probability that 2 wins with value $s$. Equation (8) follows from (6), using

$$S_1(c) = \bar{F}_2(\psi(c))(\beta_1(c) - c)$$

and rearranging, and similarly for equation (9). Note that unlike in a symmetric auction, (8) and (9) do not give bids as a function of primitives. Rather, they are part of a system of functional equations.
Because $\beta_1, \beta_2, \phi$, and $\psi$ are increasing, they are differentiable almost everywhere. If $\beta_2'(c_2) = 0$ at some $c_2 < c_1 - A$, then near $\beta_1(\phi(c_2))$, the probability of winning changes arbitrarily fast in $b_1$, and a small decrease in $b_1$ is optimal. Thus, $\beta_2'(c_2) > 0$ where it exists, and similarly for $\beta_1'$.

Differentiating $\beta_1(\phi(c)) = \beta_2(c) + A$, where $\beta_1'$ and $\beta_2'$ exist,

$$
\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0.
$$

(13)
The remaining equations are a matter of calculation and substitution.

Assume that $\phi \in C^k$, for some $k \leq k$. Then, each part of the RHS of (12) is $C^k$. Since $\phi = \int \phi'$, it follows that $\phi$ is $C^{k+1}$, and so by induction, $\phi$ is $C^{k+1}$. That $\beta_1$ and $\beta_2$ are $C^{k+1}$ then follows from (8) and (9).

5.2 Some Quick Connections Between First Price Auctions and $\rho$-Concavity.

We begin by noting a simple but useful connection between $\rho$-concavity and first price auctions.

**Proposition 3** $\beta_1'(c) = W_{S_1}(c)$, and $\beta_2'(c) = W_{S_2}(c)$

To see this, note that since $S_1(c) = \int_{c}^{c_1} \tilde{F}_2(\psi(s))ds$, $S_1'(c) = -\tilde{F}_2(\psi(c))$, and $S_2''(c) = f_2(\psi(c))\psi'(c)$. Thus,

$$
W_{S_1}(c) = \frac{S_1(c) f_2(\psi(c))}{\tilde{F}_2^2(\psi(c))} \frac{1}{\phi'(\psi(c))}.
$$

and the result follows from (10).

As an easy application of this, we have

**Proposition 4** Consider a symmetric standard FPA with $F_1 = F_2 = F$, and $A = 0$. If $f$ is increasing, then $\beta'(c) \leq \frac{1}{2}$ for all $c$. If $f$ is decreasing, then $\beta'(c) \geq \frac{1}{2}$ for all $c$.

To see this, note that $S_1(c) = S_2(c) = \int_{c}^{1} \tilde{F}(s) ds$, and so

$$
\beta'(c) = W_S(c) = W \int_{\tilde{F}}(c) = 1 - \rho \int_{\tilde{F}}(c).
$$

(14)

One way of thinking about $\beta'(c)$ is that it specifies how cost changes are shared between the seller and the buyer. By (14), this is determined precisely by the local $\rho$-concavity of $\int \tilde{F}$.
As another brief example, consider two symmetric first price auctions with cost distributions $F$ and $G$, and with equilibrium bid functions $\beta_F(c)$ and $\beta_G(c)$. Under what conditions can one say that bidding is more or less aggressive with $G$ than with $F$? A partial answer to this is as follows.

**Proposition 5** Let $F$ with support $[a,1]$ and $G$ with support $[0,1]$, $a \geq 0$ be related by $G(c) = F(\gamma(c))$, where $\gamma: [0,1] \to [a,1]$ satisfies $\gamma(c) \geq c$.

**Claim 1** (1) If $W_{f_F}$ is increasing$^{20}$ and $\gamma$ is concave then for all $c \in [a,1]$,

$$\beta'_G(c) \leq \beta'_F(c) \text{ and } \beta_G(c) \geq \beta_F(c).$$

(2) If $W_{f_F}$ is decreasing and $\gamma$ is convex then for all $c \in [a,1]$,

$$\beta'_G(c) \geq \beta'_F(c) \text{ and } \beta_G(c) \leq \beta_F(c).$$

One way of thinking about $G$ and $F$ is that one first draws a cost according to $G$, and then, to arrive at $F$, suffers a cost penalty given by $\gamma(c) - c$. Concavity (convexity) of $\gamma$ says that incremental cost savings in the original draw have lower (higher) incremental impact on final costs when costs are higher than when they are lower.$^{21}$

Note that under (1), despite the fact that costs are stochastically lower under $G$, bids for any given cost are higher, and less of any given cost saving for a bidder shows up in a more aggressive bid. (The buyer is better off under $G$ than under $F$, with the effect through better costs dominating the effect through less aggressive bidding.) Under (2), the buyer benefits both from the stochastically lower costs implied by $G$ and from more aggressive bidding.

The comparison of $\beta'_G(c)$ and $\beta'_F(c)$ is direct from Remark 3, combined with the following observation about $\rho$-concavity.

**Lemma 4** If $W_{f_F}$ is increasing (decreasing), $\gamma$ is concave (convex), and $\gamma(c) \geq c$ for all $c \in [a,1]$, then

$$W_{f_{F \circ \gamma}}(c) \geq (\leq) W_{f_F}(c)$$

for all $c \in [a,1]$.

$^{20}$We will use and interpret this condition extensively later in the paper.

$^{21}$When $\gamma$ is concave, $G$ dominates $F$ in the convex transform order, and conversely when $\gamma$ is convex. See Shaked and Shanthikumar (1994).
The comparisons of \( \beta_G(c) \) and \( \beta_F(c) \) follow given that \( \beta_G(1) = \beta_F(1) = 1 \).

As a final quick application, let us depart momentarily from the two bidder setting.

**Proposition 6** Fix \( F \) and let \( \beta_n \) be the symmetric first price equilibrium bid function with \( n \) bidders. Then \( \beta_n'(c) \) increases in \( n \), and \( \beta_n(c) \) decreases in \( n \) for all \( c \).

That \( \beta_n(c) \) is decreasing in \( n \) is already well known, as \( \beta_n(c) \) is the expectation of the lowest cost from \( n - 1 \) bidders. But, how incremental cost improvements are shared is less obvious, and, as the previous claim illustrates, the forces involved are potentially thorny. The proof relies on first observing that as for the two bidder case, the symmetric equilibrium bid with \( n \) bidders is characterized by

\[
\beta_n'(c) = W_f \bar{F}_n(c) = 1 - \rho_f \bar{F}_{n-1}(c).
\]

The result is then implied by the following property of \( \rho \)-concavity, and by the observation that \( \beta_n(1) = 1 \) for all \( n \).

**Lemma 5** Consider \( h \) decreasing and log-concave and \( \alpha > 1 \). Then

\[
\rho_{f h^\alpha}(c) \leq \rho_f h(c).
\]

In particular, to compare \( \beta_{n+1}'(c) \) and \( \beta_n'(c) \), set \( \alpha = \frac{n}{n-1} \) and \( h(c) = \bar{F}_{n-1}(c) \).

6 Ranking First and Second Price Auctions

In this section, we study the relative merits of SPBAs and FPHAs in a setting that arises naturally in procurement auctions. The analysis both has considerable economic interest, and serves as our main illustration of how the tools of \( \rho \)-concavity help us to better understand asymmetric FPHAs.

6.1 Second Price Auction Formats

In a second price bonus auction (SPBA\(A\)) the auctioneer announces a bonus \( A \), and requests sealed bids from 1 and 2. The low bidder wins.\(^{22}\) When

\(^{22}\)Ties are zero probability in equilibrium and the tie breaking rule is inessential (see Jackson and Swinkels (2005)).
1 wins, he receives $\min(b_2 + A, \bar{c}_1)$, while if 2 wins, he receives $\min(b_1, 1)$. Putting these maxima on payments guarantees that the bidders do not receive an amount above their highest possible cost.

For 2, it is weakly dominant to set $\beta_2(c_2) = c_2$, while for 1, it is weakly dominant to set $\beta_1(c_1) = c_1 - A$. Thus, 1 wins iff $A \geq c_1 - c_2$. SPBAs are easy to implement using open formats, and their equilibria are based on dominant strategies that are easy to compute relative to their FPHA counterparts. From a procurement perspective this presents often overlooked advantages.

Because $\beta_1(c_1) = c_1 - A$, 2 never wins when $c_2 > \bar{c}_1 - A$. So, without loss of generality, we restrict $A \leq \bar{c}_1$.

6.2 Relating Distributions

Our results will depend on how $F_1$ and $F_2$ are related. Define $\gamma_F$ implicitly by

$$\bar{F}_1(\gamma_F(c_2)) = \bar{F}_2(c_2).$$

Similarly, define $\gamma_T$ by

$$\frac{f_1}{F_1} \left( \gamma_T(c_2) \right) = \frac{f_2}{F_2} \left( c_2 \right),$$

and $\gamma_{T'}$ by

$$\frac{f_1'}{f_1} \left( \gamma_{T'}(c_2) \right) = \frac{f_2'}{f_2} \left( c_2 \right).$$

Finally, define $\lambda_A$ by $\lambda_A(c_2) = c_2 + A$, so that SPBA$_A$ implements precisely $\lambda_A$. Of particular interest to us will be $\lambda_T$, where $\tau = \bar{c}_1 - 1$. This is the line of slope 1 that passes through $(\bar{c}_1, 1)$.

**Assumption 1** $\gamma'_{F} \geq 1$ and $\gamma'_{T} \geq 0$.

This is automatic when $F_1 = F_2$, but otherwise imposes that $F_1$ is, in a particular sense, a stretch and convexification of $F_2$.

\footnote{Note that $\gamma_F(c_2)$ also satisfies automatically $F_1(\gamma_F(c_2)) = F_2(c_2)$.}

\footnote{As in Footnote 19, since the objects are monotone, there is no ambiguity in defining $\gamma_{T'}(c_2) = \bar{c}_1$ if $f_1' / f_1(c_1) < f_2' / f_2(c_2)$ for all $c_1$, and analogously if the inequality is reversed.}

\footnote{That is, $F_1$ dominates $F_2$ in the dispersive order. See Shaked and Shanthikumar (1994). For application of the dispersive order to auction theory, see Gauza and Penalva (2009).}

\footnote{Recall (Footnote 11) that $\bar{c}_1 < 0$ creates no interpretational difficulties.}
Assumption 2 $\rho_{F_i}$ is decreasing.

By Corollary 1, this is automatic if $f_1$ is increasing. Assumption 2 can never be satisfied if $f_1$ decreases near $\tilde{c}_1$ but $f_1(\tilde{c}_1) > 0$. For densities where $f_1(\tilde{c}_1) = 0$, the condition that $W_{F_1}$ is increasing is neither vacuous nor difficult to satisfy:

Example 1 Choose $h$ increasing and log-concave. Then, $W_H$ is increasing (Corollary 1). Setting $f_1 = H$, $W_{f_1} = W_H$ is increasing, and thus so is $W_{F_1}$.

If $h$ is decreasing with $h(1) > 0$, then $W_H(1) = \frac{1}{2}$, while (Corollary 2), $W_H(c) > \frac{1}{2}$ for $c < 1$. So, (excepting infinitely many sign changes in $W_H$), $W_H$ will be decreasing over some interval near 1. On this interval, $W_{F_1} = W_{f_1}$ is also decreasing.

Under $A_1$ and $A_2$, there is a useful ordering of the objects in play.

Lemma 6 Under $A_1$ and $A_2$, $\gamma_{\frac{1}{f'}} \geq \gamma_F \geq \gamma_{\frac{1}{f''}}$, $(\gamma_{\frac{1}{f'}})' \geq 1$, and $\gamma_F(0) \leq \gamma_{\frac{1}{f'}}(0)$.

6.3 The Optimal Mechanism

Let

$$\omega_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$$

be the virtual cost of $i$. Because $F_i$ is log-concave, $\omega' \geq 1$. Let

$$\eta(c_1, c_2) = \Delta - (c_1 - c_2) - \left( \frac{F_1(c_1)}{f_1(c_1)} - \frac{F_2(c_2)}{f_2(c_2)} \right)$$

(15)

$$= \Delta - (\omega_1(c_1) - \omega_2(c_2)).$$

---

27Since

$$1 - \rho_{F_i} = W_{F_i} = \frac{F_i(-f_i)}{f_i},$$

if $f$ is decreasing on $(\hat{c}, \tilde{c}_1)$, then $W_F > 0$ on $(\hat{c}, \tilde{c}_1)$. If $f(\tilde{c}_1) > 0$, but $f'(\tilde{c}_1)$ is finite, then, since $F_i(\tilde{c}_1) = 0$, $W_{F_i}(\tilde{c}_1) = 0$.

28It is worth observing that while log concavity is the conventional assumption here (partly on the basis that it is interpretable), the condition for $\omega' > 0$ is precisely $\rho_{F_i} > -1$, i.e., that $\frac{1}{f_i}$ is convex.
Consider any deterministic mechanism $\mu$ in which the buyer always buys. From incentive compatibility, $\mu$ is completely characterized by an increasing function $\phi_\mu$ such that 1 wins if and only if $c_1 < \phi_\mu(c_2)$. For any increasing function $\sigma : [0, 1] \to [c_1, c_2]$, define

$$BS(\sigma) = v_2 - 1 + \int \int I_{\{c_1 < \sigma(c_2)\}} \eta(c_1, c_2) f_1(c_1) f_2(c_2) \, dc_1 dc_2.$$  

(16)

Following Myerson (1981) or Riley and Samuelson (1981), we have:

**Lemma 7** *The buyer’s surplus from mechanism $\mu$ is $BS(\phi_\mu)$.*

This is intuitive. Always buying from 2 gives the buyer surplus $v_2 - 1$, since 2 must receive 1 if he is to sell for all $c_2$. The second term represents the change in buyer surplus from buying from 1 according to $\phi_\mu$.

From Lemma 7, it follows directly that

**Corollary 3** *Among mechanisms that always buy, 1 optimally wins if*

$$\Delta > \omega_1(c_1) - \omega_2(c_2)$$  

(17)

and 2 wins otherwise.\(^{29}\)

Thus, 1 wins if his virtual cost is no more than $\Delta$ above 2’s. This follows since by (17), $I_{\{c_1 < \phi_\mu(c_2)\}} \eta(c_1, c_2)$ is maximized point-wise. Since $\omega$ is increasing, the allocation rule is monotone and hence incentive compatible.

Define the *Optimal Allocation*, $\phi_M$, (M mnemonic for Myerson) by

$$\Delta = \omega_1(\phi_M(c_2)) - \omega_2(c_2),$$  

(18)

or, equivalently, by

$$\eta(\phi_M(c_2), c_2) = 0,$$  

(19)

so that 1 wins in the optimal mechanism for $c_1 < \phi_M(c_2)$ and 2 wins for $c_1 > \phi_M(c_2).\(^{30}\)

To tie $\phi_M$ down further, we need one more assumption.

**Assumption 3** $\rho_{F_1}$ is increasing.

\(^{29}\)Whether or not exclusion is optimal, the constrained mechanism serves its purpose in helping us compare the FPHA and SPBA.

\(^{30}\)This is well defined since $\omega$ is monotonic.

\(^{31}\)Note that $F_1(c) = F_2(c) = c^K$ has, for each, $\Delta \phi_M$ has slope 1, and can thus be implemented by an appropriately chosen SPBA.
Example 2 Let $\alpha \geq 1$. Assume $F_1(c) = F_2(c) = c^\alpha$. Then, $\rho_{F_1} = \frac{1}{\alpha}$ and so $A_3$ is satisfied. As $f' > 0$, $A_2$ is satisfied by Corollary 1, and $A_1$ is automatic as $F_1 = F_2$.

Lemma 8 Under $A_1$ and $A_3$, $\phi_M$ crosses $\gamma_F$ at most once on $[0,1)$, and, if it crosses, does so from above. Anywhere that $\phi_M(c) \geq \gamma_F(c)$, $\phi_M'(c) \leq 1$.

The heart of the proof lies in the observation that

$$\phi'_M(c_2) = \frac{1 + \rho_{F_2}(c_2)}{1 + \rho_{F_1}(\phi_M(c_2))}. \quad (20)$$

We show that under $A_1$, on $\gamma_F$, $\rho_{F_1} \geq \rho_{F_2}$, and so if $\phi_M(c_2) \geq \gamma_F(c_2)$, then $\phi'_M(c_2) \leq 1$. Since $\gamma'_F(c_2) \geq 1$, any crossing is thus from above, and unique.

6.4 Statement and Intuition for the Main Result with Increasing Densities

Consider mechanisms $\mu_1$ and $\mu_2$. Say that $\mu_1$ dominates $\mu_2$ if the $\mu_1$ and $\mu_M$ specify the same allocation on a superset of the set over which $\mu_2$ and $\mu_M$ do. From Lemma 7, $\mu_1$ thus gives the buyer higher surplus.

We will first state the result and provide intuition for the case of increasing densities. Then we will discuss the extra structure needed to deal with decreasing densities, and finally state and provide intuition for the case of decreasing densities.

Theorem 3 Assume $A_1$ and $A_3$, that $f_1$ is increasing, and that $\eta(\bar{c}_1, 1) > 0$. Then, for any $A$, there is $\hat{A}$ such that $SPBA_{\hat{A}}$ dominates $FPHA_A$.

The assumption $\eta(\bar{c}_1, 1) > 0$ says that $\Delta$ is large enough that when both players have their worst cost types, the optimal mechanism gives the job to 1. This is automatic for $\Delta > 0$ when $F_1 = F_2$, and otherwise requires that $\Delta$ be sufficiently large compared to the asymmetry between $F_1$ and $F_2$. The statement of the theorem omits $A_2$ because (by Corollary 1) it is implied by $f_1$ increasing.

The proof of Theorem 3, which unfolds over the next subsections, is essentially geometric. See Fig. 1. First, we will use Lemma 8 to show that $\eta(\bar{c}_1, 1) > 0$ implies that $\phi'_M \leq 1$. So, $\phi_M$ lies above $\lambda_\tau$. We use this to show that for any $A \leq \tau$, $FPHA_A$ is dominated by $SPHA_\tau$. The heart of the proof is to show that for $A > \tau$, $\phi'_{FP}(c_2) \geq 1$ over all relevant ranges. But then, a SPBA mechanism which implements a line of slope 1 through the unique crossing point of $\phi_M(c_2)$ and $\phi_{FP}(c_2)$ dominates $FPHA_A$. This is $\lambda_{\hat{A}}$ as illustrated in Fig. 1.
6.5 Two Complexities with Decreasing Densities

Now, let’s turn to densities where \( f_1(c_1) = 0. \) We need to surmount two hurdles. First, we will again want to show that for any \( A \) that is large enough not to be easily ruled out as suboptimal, \( \phi_{FP} \geq 1 \) over all relevant ranges. To do so, we need the following replacement for the condition that \( f_1 \) is increasing.

**Assumption 4** \( \left( \frac{\log f_1}{c} \right)^{\prime\prime} (c) \) is well-defined at \( \bar{c}_1 \) and reaches a minimum at \( \bar{c}_1. \)

Assumption 4 requires that in a sense, the concavity of \( \log f \) versus that of \( \log F \) is at its smallest at \( \bar{c}_1. \) This is satisfied broadly but not universally in examples we have checked.

**Example 3** Consider \( F_1(c) = F_2(c) = 1 - (1 - c)^{a-1} \). It can be checked that \( W_{F_1}(c) = \frac{a-1}{a} \) and so \( A2 \) is trivially satisfied. Since \( f_1(c) \) is decreasing, \( A3 \) is satisfied because by log-concavity

\[
W_F = - \left( - \frac{f'}{f} \right) \left( \frac{F}{f} \right)
\]

\(^{32}\)Recall that \( A2 \) is impossible if \( f_1 \) neither increases nor goes to zero.
is the negative of the product of increasing positive functions. Finally, $\log F_1(c) = \alpha \ln (1 - c)$ while $\log f_1(c) = \ln \alpha + (\alpha - 1) \ln (1 - c)$ and thus

$$
\frac{(\log f_1)^\prime\prime}{(\log F_1)^\prime\prime}(c)
$$

is constant, satisfying $A_4$.

**Lemma 9** Let $H$ satisfy $A_2$ and $A_4$, and let $F_1$ be given by $F_1 = H^n$, for some $n \in (0, \infty)$. Then, $F_1$ satisfies $A_2$ and $A_4$ as well.

When $n$ is an integer greater than one, $F$ is the result of taking the most favorable from $n$ draws from $H$, while when $\frac{1}{n}$ is an integer, $H$ can be thought of as the most favorable from $n$ draws from $F$.

**Remark 1** Lemma 9 provides a ready source of both hump-shaped and decreasing examples. Choose $h$ increasing and log-concave with $W_h(c) =$ finite. Then, $W_H$ is negative and increasing (Corollary 1), so that $A_2$ is satisfied, and $A_4$ is automatic.$^{33}$ If $F_1 = H^n$, then

$$
\frac{f'}{f} = - (n - 1) \frac{h}{H} + \frac{h'}{h}
$$

Since $W_H(c_1) = 0$, for any $n > 1$, $f' < 0$ near $c_1$. For $n$ near enough $1$, $f$ will thus be hump-shaped, while for $n$ large enough, $f$ will be decreasing.

**Remark 2** Choose $h$ increasing and log-concave. Setting $f = H$, we have that $W_f = W_H$ is increasing, and thus so is $W_{\bar{F}}$. This construction is thus also a ready source for decreasing densities which satisfy $A_2$. It is easy to check that densities constructed this way satisfy $A_4$ as well.

By judicious choice of $\Delta$, one can start from these symmetric distributions that satisfy the conditions, and distort one or the other of them via $\gamma_F$ in such a way as to continue to satisfy them.

To see the second issue, note first that when $f_1(c_1) = 0$, the result that $\phi_M'$ is everywhere less than one fails:

$$
\frac{(\log h)^\prime\prime}{(\log H)^\prime\prime}(c) = W_H^2(c) \left( \frac{1 - W_h(c)}{1 - W_H(c)} \right)
$$

which is weakly positive by log-concavity. Since $h$ is increasing, $W_H(c) = 0$, while $W_h(c)$ is finite by assumption, and so the RHS tends to 0.

$^{33}$By Lemma 20.
Lemma 10 Assume A1, that $f_1(\bar{c}_1) = 0$, and that $\gamma_{\bar{F}}$ is not the identity. Then, for any finite $\Delta$, $\phi_M(1) = \bar{c}_1$. Further, $\phi_M$ crosses $\gamma_F$ at some $c_\Delta \in (0,1)$. As $\Delta$ increases, so does $c_\Delta$, and for $\Delta \to \infty$, $c_\Delta \to 1$. On some interval near 1, $\phi_M'(c) > 1$.

The key is that virtual costs diverge as costs approach their worst case. But, given that near 1, $\gamma_{\bar{F}}' > 1$, (as $\gamma_{\bar{F}}$ is not the identity) they diverge faster for player 1, eventually swamping any given $\Delta$.

We thus have the possibility illustrated in Fig. 2. For clarity, we have zoomed in on the top right corner of the picture, starting at some cost $K$ for 2 and $\phi(K)$ for 1. Here, the SPBA through the intersection of $\phi_M$ and $\phi_{FP}$ does not dominate $\phi_{FP}$, because on the shaded triangular region in the upper-right corner, $\lambda_{\bar{A}}$ is getting things wrong, while $\phi_{FP}$ is getting things right.\footnote{Since virtual costs explode in this case, some exclusion is presumably optimal even in first and second price mechanisms. We leave the question of the effects of such exclusion for future research.}

The key is that $\lambda_{\tau}$ no longer lies everywhere above $\phi_M$. It can be shown, however, that as either $\Delta$ grows, or $F_1$ and $F_2$ become close to each other, the region over which $\lambda_{\tau}$ is above $\phi_M$ becomes arbitrarily small, so that Fig. 2 represents a smaller and smaller corner of the cost space.

To attain a result in this setting, we will first have to relax our notion of one mechanism being better than another from dominance to simple ex-ante
superiority. We will also need a different form of our assumption that $\Delta$ is not too small relative to the asymmetry between $F_1$ and $F_2$. To do this, define $c_\tau$ as the (unique) point at which $\phi_M$ crosses $\lambda_\tau$, and consider the line $\delta_\tau$ given by

$$
\delta_\tau(c) = \begin{cases} 
\lambda_\tau(c) & \text{for all } c \leq c_\tau \\
\phi_M(c) & \text{for all } c \geq c_\tau 
\end{cases}.
$$

Exactly why $\delta_\tau$ is the right object to think about will become clear shortly. It is illustrated in Fig. 4.

**Assumption 5** There exists $\hat{A}$ such that $BS(\lambda_\hat{A}) \geq BS(\delta_\tau)$.

That is, there is a second price auction that does at least as well, in ex-ante terms, as $\delta_\tau$. This assumption can indeed be interpreted as saying that $\Delta$ is not too small relative to the asymmetry between $F_1$ and $F_2$. As discussed, as $\Delta$ grows, or the asymmetry between $F_1$ and $F_2$ shrinks, $\delta_\tau$ converges to $\lambda_\tau$, and so there is a small area on which $\delta_\tau$ is doing something superior to $\lambda_\tau$, and a considerable area on which $\lambda_\tau$ is misallocating in favor of player 2.  

### 6.6 Statement and Intuition for the Main Result with Decreasing Densities

We are now ready to state a counterpart to Theorem 3.

**Theorem 4** Assume A1-A5. Then, the ex-ante optimal SPBA gives the buyer higher expected surplus than any FPHA.

The proof relies on many of the same ideas as that of Theorem 3. As before, $A \leq \tau$ do not make sense. Once again, we can show that for $A > \tau$, $\phi'_M > 1$. We use this to derive the crossing properties of $\phi_M$ and $\phi_{FP}$, and show how to find a superior SPBA for any given FPHA.

**Corollary 4** Let $F_1 = F_2 = F$ and $\Delta > 0$. Assume that $F$ satisfies A2-A4. Then the optimal FPHA is dominated by an appropriately chosen SPBA.

---

\[35\text{In particular, it can be shown that for any } \varepsilon > 0, \text{ there is } \Delta \text{ large enough that } BS(\lambda_{\tau+\varepsilon}) > BS(\delta_\tau).\]
To see this note that in this case, A1 is clearly satisfied and $\phi_M$ lies strictly above $\lambda_F = \gamma_F$ (the main diagonal) on $[0,1)$. Thus any FPHA$_A$ with $A < 0$ is dominated by FPHA$_0$. FPHA$_0$ cannot be optimal since small $A > 0$ distort away from the optimum on a small set of high costs, but improve the allocation on a substantially bigger set. And, since $\phi_M$ lies above $\gamma_F$, $\phi'_M \leq 1$, and so a SPHA through the intersection of $\phi_{FP}$ and $\phi_M$ is dominant as it was for Theorem 3.

6.7 Interpreting the Results

Under the conditions of either of these theorems, for any given first price handicap mechanism, there is a superior second price mechanism. Our conditions are sufficient for the dominance result, but far from necessary. In Mares and Swinkels (2009b), we show how to extend the result to a setting in which there are several players drawing according to each of two distributions. We provide a numerical technique for calculating equilibria of two player asymmetric first price auctions, and use it to show that the dominance result continues to hold in a wide variety of numerical examples that fail the conditions of our theorems. In particular, we have no known counterexample to $\phi'_{FP} > 1$ (despite considerable effort to find one). In Mares and Swinkels (2009c) we show how bounds on $\rho_F$ translate into recommendations for second price handicaps that do not depend on further details of $F$, but guarantee remarkably good performance. Those bounds are strongest in precisely the case (concave virtual costs) where our results here about the dominance of a first price mechanism do not directly apply (because $\phi_M$ has slope greater than 1, and so our “crossing” argument does not apply). So, in such settings the SPBA is in fact close to optimal, and a good such SPBA can be found knowing only a minimal amount about the distribution over costs.

We thus think that there is a substantial presumption that firms using FPHA’s (which are fairly common) should consider SPBA’s instead.\footnote{In Mares and Swinkels (2009a), we study further properties of second price and optimal auctions in our setting, focusing, for example, on the comparative statics of the optimal handicap with respect to changes in the distribution, and giving a finer grained analysis of how properties of the distribution feed into the shape of the optimal allocation.}

- First, our results suggest that they are likely to provide higher surplus to the buyer.

- Second, the search for the optimal SPBA should be much simpler for firms than the search for the optimal FPHA, and our results strongly
suggest that a firm loses nothing by discarding all FPHA’s out of hand, and focusing its resources on finding the best SPBA instead.\footnote{Mares and Swinkels (2009a) explore various properties of the optimal bonus in a \textit{SPBA}.}

- Third, SPBA’s are much simpler for firms to bid in, and so participation may be enhanced.

- Fourth, predictions by the firm about behavior in SPBAs are likely to be more robust than the somewhat tenuous belief that bidders will find their way to the equilibrium of the wildly challenging asymmetric first price auction. One could, of course, hope that confused bidders make mistakes that are systematically to the good of the firm, but at least on an intuitive level, one could imagine bidders in complicated settings reasoning along the lines of the winner’s curse that the news that they have won is strongly suggestive that they have misunderstood the situation and bid too much.\footnote{One of the authors of this work was offered the opportunity to help a major industrial firm design new and more complicated mechanisms with precisely this goal! The job was turned down on a number of grounds, not least of which was a belief that this unlikely to be a great strategy for the firm involved.}

- Finally, once bidders are familiar with second price bonus mechanisms with constant bonuses, second price mechanisms with somewhat more complicated bonuses may be feasible,\footnote{Again subject to various legal and other restrictions.} which may allow one to get quite close to $\phi_M$. In contrast, no matter how complicated is the handicap function in a first price mechanism, it will always be the case that, at a minimum, it undoes all reallocation at low costs, and so must always be substantially distant from optimal.

Industry practice is routinely at a variance to the advice we give here, even for firms where the simplifications inherent in our model seem a reasonable approximation to reality. We think it probable that many firms could improve their practices.

### 6.8 Proof of Theorems 3 and 4

#### 6.8.1 Small $A$ are not Optimal

First, we derive coarse but effective bounds on $\phi_{FP}$.
Lemma 11 Anywhere that $\phi_{FP}(c_2) \leq \min \left( \frac{\gamma_F}{\partial} (c_2), c_2 + A \right)$, $\phi'_{FP}(c_2) \geq 1$. Anywhere that $\phi_{FP}(c_2) \geq \max \left( \frac{\gamma_F}{\partial} (c_2), c_2 + A \right)$, $\phi'_{FP}(c_2) \leq 1$. Each inequality is strict unless $\phi_{FP}(c_2) = \frac{\gamma_F}{\partial} (c_2) = c_2 + A$.

Proof From Theorem 2, for any $c_2 > 0$ such that $c_1 = \phi_{FP}(c_2)$ we have

$$
\phi'_{FP}(c_2) = \left( \frac{S_1(c_1)}{S_2(c_2)} \right) \left( \frac{f_2(c_2)}{F_2(c_2)} \right) \left( \frac{S_2(c_2)}{S_1(c_1)} \right) \left( \frac{F_2(c_2)}{F_1(c_1)} \right) = \left( \frac{\beta_1(c_1) - c_1}{\beta_2(c_2) - c_2} \right) \left( \frac{f_2(c_2)}{F_2(c_2)} \right) \left( \frac{f_1(c_1)}{F_1(c_1)} \right) = \left( 1 + \frac{A - (c_1 - c_2)}{\beta_2(c_2) - c_2} \right) \left( \frac{f_2(c_2)}{F_2(c_2)} \right) \left( \frac{f_1(c_1)}{F_1(c_1)} \right),
$$

(21)
since $\beta_1(c_1) = \beta_2(c_2) + A$, by definition of $\phi_{FP}$.

Thus, when $\phi_{FP}(c_2) \leq \min \left( \frac{\gamma_F}{\partial} (c_2), c_2 + A \right)$, each term in (21) at least 1, while when $\phi_{FP}(c_2) \geq \max \left( \frac{\gamma_F}{\partial} (c_2), c_2 + A \right)$, each term is at most 1. ■

For the case $f$ increasing, define $c_M$ as the first point at which $\phi_{M}(c_2) = \tilde{c}_1$. Since $\eta(\tilde{c}_1, 1) > 0$, $c_M < 1$.

Lemma 12 Under the conditions of Theorem 3, $\phi_M \geq \lambda_{\tilde{c}_1 - c_M}$.

This follows since from $c_M < 1$, and from Lemma 8, $\phi_M$ is everywhere above $\gamma_F$, and so, has $\phi'_M \leq 1$.

We can now complete our proof that $A \leq \tau$ do not make sense.

Lemma 13 Under the conditions of either Theorem 3 if $A \leq \tau$, then $FPHA_A$ is dominated by $SPBA_{\tilde{c}_1 - c_M}$.

This follows from Lemma 11, since $\phi_{FP}$ starts below $\lambda_{\tau}$, it cannot get above it. But then, since $\tilde{c}_1 - c_M > \tau = \tilde{c}_1 - 1$, and $\phi_M \geq \lambda_{\tilde{c}_1 - c_M}$, $SPBA_{\tilde{c}_1 - c_M}$ dominates $FPHA$.

Lemma 14 Under the conditions of Theorem 4, if $A \leq \tau$, then $FPHA_A$ is dominated by $\delta_{\tau}$.

The proof follows again because $\phi_{FP}$ starts below $\lambda_{\tau}$ and so cannot get above it. Subject to never being above $\lambda_{\tau}$, $\delta_{\tau}$ is the best allocation available.
6.8.2 A Boundary Condition

Given Lemmas 13 and 14, we can restrict attention to \( A > \tau \). For such \( A \), we aim to show that \( \phi'_{FP} > 1 \) everywhere on \((0, \bar{c}_{1} - A)\). In this section, we show that this is true at the two boundaries of the domain of \( \phi_{FP} \), i.e., at 0 and at \( \bar{c}_{1} - A \).

**Lemma 15** Under \( A > 1 \), if \( A > \tau \), then \( \phi'_{FP}(0) > 1 \).

**Proof** Note first that since \( \tau = \bar{c}_{1} - c_{M} \geq \bar{c}_{1} - 1, \bar{c}_{1} - A < 1 \). Since \( \gamma_{F} \geq 1 \)

\[
\gamma_{F}(1) - \gamma_{F}(0) \geq 1,
\]

and so

\[
\gamma_{F}(1) - 1 \geq \gamma_{F}(0)
\]

or

\[
\bar{c}_{1} - 1 \geq \gamma_{F}(0)
\]

and so

\[
A > \bar{c}_{1} - 1 \geq \gamma_{F}(0) = \xi_{1} = \gamma_{F}(0).
\]

Lemma 11 thus applies at 0 since \( \phi(0) = \xi_{1} = \gamma_{F}(0) \leq \gamma_{F}(0) \).

**Lemma 16** Under \( A > 1 \), if \( A > \tau \), then \( \limsup_{c \to \bar{c}_{1} - A} \phi'(c) = \infty \), and \( \liminf_{c \to \bar{c}_{1} - A} \phi'(c) > 1 \).

So, (modulo the annoying possibility of a discontinuity of the second type in \( \phi' \) at \( \bar{c}_{1} - A \)), \( \phi' \) becomes arbitrarily large as \( c \to \bar{c}_{1} - A \). The proof of this is surprisingly involved. But, the key driver is that since \( \bar{c}_{1} - A < 1 \), the behavior of \( S_{1} \) and \( S_{2} \) at \( \xi_{1} \) and \( \bar{c}_{1} - A \) are very different, with, in particular, the surplus of player 1 changing much more quickly. \(^{40}\)

6.8.3 Interior Minima

Given Lemmas 15 and 16, if \( \phi'_{FP} \leq 1 \) anywhere on \([0, c_{1} - A]\), then \( \phi'_{FP} \) achieves a global minimum at some \( r \in (0, c_{1} - A) \) with \( \phi'_{FP}(r) \leq 1 \). Given that \( \phi_{FP} \) is \( C^{2} \), and that \( r \) is interior, \( \phi''_{FP}(r) = 0 \). To complete our proof of Theorem 2, it remains only to show that the two conditions \( \phi'_{FP}(r) \leq 1 \) and \( \phi''_{FP}(r) = 0 \) lead to a contradiction.

The first essential step in doing this is the following proposition.

\(^{40}\)Unlike much of the development to date, this result depends crucially on \( A > \tau \).
Proposition 7 Assume that for some \( c \in (0, c_1 - A) \), \( \phi'_{FP}(c) \leq 1 \). Then,

\[
\phi''_{FP}(c) \geq \left( \frac{1}{S_1(\phi_{FP}(c))} \frac{f_2}{f_2'} (c) - 2 \right) \left( \frac{f_1}{f_1'} (\phi_{FP}(c)) - \frac{f_2}{f_2'} (c) \right) + \left( \frac{f_2'}{f_2} (c) - \frac{f_1'}{f_1} (\phi_{FP}(c)) \right),
\]

where \( \geq \) denotes that the LHS is positive whenever the RHS is.

Proof Since

\[
\phi'_c = \frac{S_1(\phi_{FP}(c)) f_2}{S_2(c) \frac{f_1}{f_1'} (\phi_{FP}(c))} = \frac{T}{B},
\]

an expression for \( \frac{\phi''_{FP}(c)}{\phi_{FP}(c)} = (\log \phi'_{FP}(c))' \) is

\[
\phi'_c = \frac{S_1(\phi_{FP}(c)) f_2}{S_2(c) \frac{f_1}{f_1'} (\phi_{FP}(c))} = \left( \log \frac{f_2}{f_2'} \right)' = \left( \log \frac{1}{f_2} + \log \frac{f_2}{f_2'} \right)' > 0,
\]

and so,

\[
\left( \frac{f_2}{f_2'} + 2 \frac{f_2}{F_2} \right) (c) \geq \phi'_{FP}(c) \left( \frac{f_2}{f_2'} + 2 \frac{f_2}{F_2} \right) (c).
\]

Also, from (6),

\[
S_1(\phi_{FP}(c)) = -F_2(c),
\]

and so

\[
\phi'_c S'_1(\phi_{FP}(c)) = \phi_{FP}(c) \left( \frac{f_2}{f_1'} (\phi_{FP}(c)) \right) = \phi_{FP}(c) \left( \frac{f_2}{f_2'} \right) = \frac{\phi_{FP}(c)}{T}
\]

and similarly

\[
\frac{S'_2}{S_2} = \frac{\frac{f_2}{f_1} (\phi_{FP}(c))}{B} = \phi_{FP}(c) \frac{\frac{f_2}{f_1} (\phi_{FP}(c))}{T}.
\]

Substituting (25), (26), and (27) into (24), collecting terms, and cancelling \( \phi'_{FP}(c) > 0 \), we are done. ■
The expression in Proposition 7 involves the object \( S_1 (\phi_{FP} (c)) \), which, as a function of the entire equilibrium to the right of \( \phi_{FP} (c) \), is inherently forbidding. The following lemma is very helpful in this regard.

**Lemma 17** \( S_1 (\phi_{FP} (r)) \frac{f_2 (r)}{f_2 ^2 (r)} < W_{F_1} (\phi_{FP} (r)) \).

**Proof** By a change of variables,

\[
S_2 (r) = \int_r ^{\bar{c}_1 - A} \tilde{F}_1 (\phi_{FP} (s)) ds \\
= \int_{\phi_{FP} (r)} ^{\phi_{FP} (\bar{c}_1 - A)} \tilde{F}_1 (s) \psi' (s) ds \\
< \frac{1}{\phi'_{FP} (r)} \int_{\phi_{FP} (r)} ^{\bar{c}_1} \tilde{F}_1 (s) ds,
\]

where the strict inequality follows since \( \phi'_{FP} \) is continuous and \( \lim \sup \phi'_{FP} = \infty \). Multiplying both sides by \( \frac{f_1}{f_2} (\phi_{FP} (c)) \) yields

\[
S_2 (r) \frac{f_1}{f_2} (\phi_{FP} (c)) < \frac{1}{\phi'_{FP} (r)} W_{F_1} (\phi_{FP} (c)). \tag{28}
\]

But since \( T = \phi'_{FP} (r) B \), we are done. ■

Putting these together yields our key building block. Let

\[
H (c_1, c_2) = \left( \frac{1}{W_{F_1} (c_1)} - 2 \right) \left( \frac{f_1}{F_1} (c_1) - \frac{f_2}{F_2} (c_2) \right) + \frac{f_1'}{f_2} (c_2) - \frac{f_1'}{f_1} (c_1). \tag{29}
\]

**Proposition 8** Assume that \( \frac{f_1}{F_1} (\phi_{FP} (r)) - \frac{f_2 (r)}{F_2 (r)} \geq 0 \). Then,

\[
\phi''_{FP} (r) >_s H (\phi_{FP} (r), r) . \tag{30}
\]

**Proof** This is immediate from Lemma 17 and Proposition 7. ■

It is potentially of interest for further work that nothing in the development of Proposition 8 depended on \( A1 - A5 \). Two more small lemmas help us along.

**Lemma 18** \( \phi_{FP} (r) \leq r + A \).
Proof To see this, note that since $\gamma_A$ lies weakly above $\gamma_F$ it follows from Lemma 11 that if $\phi_{FP}(c_2) \geq \gamma_A(c_2)$, then $\phi_{FP}(c_2) \leq 1$. As $\gamma_A = 1$, and since $\phi_{FP}(0) = \gamma_F(0) \leq \gamma_A(0)$, $\phi_{FP}$ cannot get above $\gamma_A$. ■

Lemma 19 \( \frac{f_1'}{F_1'}(\phi_{FP}(r)) - \frac{f_2'}{F_2'}(r) \geq 0 \).

Proof This follows from Lemma 11, since by Lemma 18, \( 1 + \frac{A-(\phi_{FP}(r)-r)}{f_2'(r)-r} \geq 1 \), and so, as $\phi_{FP}'(r) \leq 1$, it must be that \( \frac{f_2'(r)}{f_1'(\phi_{FP}(r))} \leq 1 \). ■

Completion of the proof of Theorem 3 Note that since $f_1$ is increasing, by Corollary 2,

\[
\frac{1}{W_{F}(r)} - 2 \geq 0,
\]

while by Lemma 6 and Lemma 19, \( \frac{f_1'(r)}{f_2'(r)} - \frac{f_1'}{F_1'}(\phi_{FP}(r)) \geq 0 \). But then, each term of $H(\phi_{FP}(r), r)$ is non-negative, and so by Proposition 8, $\phi_{FP}''(r) > 0$, a contradiction.

Thus, $\phi_{FP} > 1$, and so since $\phi_M' \leq 1$, a SPBA defined by their unique crossing dominates $\phi_{FP}$. ■

Completion of the proof of Theorem 4 We first need a lemma.

Lemma 20 If \( \frac{(\log f_1)''}{\log F_1}'(\bar{c}_1) \) exists, then

\[
\frac{(\log f_1)''}{\log F_1}'(\bar{c}_1) = W_F(\bar{c}_1).
\]

This in hand, define

\[
\hat{H}(t) = \left( \frac{1}{W_f F_1(\phi_{FP}(r))} - 2 \right) \left( \frac{f_1'}{F_1'}(t) - \frac{f_2'}{F_2'}(r) \right) + \left( \frac{f_2'}{f_2'(r)} - \frac{f_1'}{f_1'(t)} \right)
\]

so that

\[
H(\phi_{FP}(r), r) = \hat{H}(\phi_{FP}(r)) = \hat{H}(\gamma_F'(r)) = \int_{\gamma_F'(r)}^{\phi_{FP}(r)} \hat{H}(t)dt.
\]

As before, by Proposition 8, we have a contradiction to $\phi'(r) \leq 1$ if $H(\phi_{FP}(r), r) \geq 0$. But

\[
\hat{H}(\gamma_F'(r)) = \left( \frac{f_2'}{f_2'(r)} - \frac{f_1'}{f_1'(t)} \right) \geq 0,
\]

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since on $\gamma_{F}$ the first term disappears, and by assumption we have $\gamma_{F}(r) \geq \gamma_{F}(r)$. Since by Lemma 19, $\phi_{FP}(r) \geq \gamma_{F}(r)$, it would thus be enough to show that $H'(t) \geq 0$. But,

$$H'(t) = \left( \frac{1}{W_{F_{1}}(\phi_{FP}(r))} - 2 \right) \left( \frac{f_{1}}{F_{1}} \right)'(t) + \left( - \frac{f_{1}'}{f_{1}} \right)'(t)$$

$$= s \left( \frac{1}{W_{F_{1}}(\phi_{FP}(r))} - 2 \right) + \frac{(- f_{1}')'}{f_{1}}(t)$$

$$\geq \left( \frac{1}{W_{F_{1}}(\gamma(1))} - 2 \right) + \frac{(- f_{1}')'}{f_{1}}(\gamma(1))$$

$$= 0,$$

where the inequality follows since $\frac{(- f_{1}')'}{f_{1}}(t)$ is minimized at $\gamma(1)$ and since $W_{F_{1}}$ is increasing, and the last line follows by Corollary 1, and so we again have a contradiction. Hence, as before, we have $\phi_{FP}' > 1$.

Consider the first crossing point $c_{2}^*$ of $\phi_{FP}$ and $\phi_{M}$, and let $\tau^* = \phi_{M}(c_{2}^*) - c_{2}^*$. Consider first the case $c_{2}^* \leq c_{r}$, illustrated in Fig. 3. Note that $\tau^* \geq \tau$, and so, to the right of $c_{2}^*$, $\lambda_{\tau^*}$ lies above $\phi_{M}$ (since between $c_{2}^*$ and $c_{r}$, $\phi_{M}' \leq 1$, while after $c_{r}$, $\phi_{M} \leq \lambda_{\tau^*}$. To the left of $c_{2}^*$, $\phi_{M}' \leq 1$. So, $SPB\lambda_{\tau^*}$ dominates $\phi_{FP}$.

If $c_{2}^* > c_{r}$, then everywhere before $c_{r}$, $\phi_{M} < \lambda_{\tau} \leq \phi_{M}$. Since $\delta_{\tau}$ coincides with $\phi_{M}$ to the right of $c_{r}$, it thus follows that $\delta_{\tau}$ dominates $\phi_{M}$.

We are thus done since by A5, there exists $\hat{A}$ such that $BS(\lambda_{\hat{A}}) \geq BS(\delta_{\tau}).$\footnote{Note that this is the only place in the development were we resort to the weaker notion of ex-ante surplus versus dominance.}

7 Our Ranking Results in Context

Maskin and Riley (2000a) analyze auctions with a seller and two buyers with asymmetrically distributed values. In one specification of their model, a “weak” buyer draws his value from $F$ with support $[0, 1]$ and a “strong” buyer draws his value from $F_{s}$ with support $[s, s + 1]$ given by

$$F_{s}(x) = F(x + s).$$
Figure 3: Illustration of Theorem 4 when $c_2^* \leq c_\tau$.

Figure 4: Illustration of $\delta_\tau$ (thick line) when $c_\tau < c_2^*$.
When $s$ is large, in the standard first price auction the strong bidder bids 1 in equilibrium and always wins. In the second price auction, the strong bidder will still always win but pays the value of his opponent. So, the first price auction raises more revenue than the second price auction. Maskin and Riley extend this revenue ranking to settings where $s$ is smaller, and so there is a range of bids over which the allocation is competitive.\footnote{42}{To get this result, they assume that $F$ is convex ($f$ is increasing).}

As the first step to translating this into our setting, let player 1 be the “strong” bidder, with costs distributed on $[-s, 1 - s]$ according to $F_1(c) = F_2(c + s)$. The Maskin and Riley result says that in this auction, a symmetric first price mechanism is better than a symmetric second price mechanism. But now, by setting $A = \Delta = s$, one can translate this into an auction with symmetrically distributed costs but a $\Delta$ value advantage and handicap for 1. So, running a second price mechanism with handicap $A = \Delta$ is worse than running a first price mechanism with handicap $A = \Delta$. If one is going to run an RFP process, then it is better to do so in a first price rather than second price (or equivalently open) manner.

With $f$ increasing, and $F_1 = F_2$, we are, if virtual costs are convex, in the world of Theorem 3, where we show that second price mechanisms are better than first. So, our results, at first glance, seem contradictory! The key to the resolution is to note that what Maskin and Riley are effectively doing is comparing the first and second price auction for a fixed $A$, and in particular for $A = \Delta$. While natural, this turns out to be a pretty bad choice for the auctioneer, especially in the second price case: it can be shown that $\phi_M$ lies strictly below $\lambda_\Delta$ (see Mares and Swinkels 2009b) and so the second price auction with $A = \Delta$ universally distorts too far in favor of 1. In the FPHA with $A = \Delta$, $\phi_{FP}(c) - c$ is sometimes above, and sometimes below $\lambda_\Delta$, and so does not always distort too far, providing one intuition for their ranking.\footnote{43}{We limit bidders to receive at most his highest possible cost. It is an open question as to the degree to which Maskin and Riley’s ranking depends on the absence of a reserve price in their framework.}

In contrast, what we show in this setting is that for any given value advantage, $\Delta$, and FPHA$_A$ there is a bonus $\tilde{A}$ such that the SPBA$_{\tilde{A}}$ does better (in the very strong sense of ex-post dominance). So, while Maskin and Riley show that between a focal pair of asymmetric auctions one prefers the first price mechanism, we show that if one can choose which handicap to offer, one will prefer a second price mechanism.

Translating our results back into the setting of Maskin and Riley, if one bidder has a value distribution which is a favorable shift of the other, and
one must run a mechanism which treats them the same, then choose a first price auction. But, if one is allowed to choose by how much to favor a bidder, then for anything one can do with a first price auction, one can find a second price auction (where the strong bidder is required to pay a premium when he wins) that does strictly better for the auctioneer.

Introducing $A$ also allows us to think about the Example 2 in Maskin and Riley, in which the “strong” buyer has values which are a linear stretch around 0 of those of the “weak” buyer, and the first price auction is superior. Our class of asymmetric auctions consists of those where $F_1$ can be viewed as a convexification and stretch of $F_2$, and so includes this case. Here again, we have the “contradictory” result that when handicaps can be chosen, a second price mechanism is preferred.\footnote{The general “stretch” used by Maskin and Riley is somewhat different than ours, but coincides for this example.}

### 7.1 Comparing the FPHA with the symmetric FPA

A key step in proving Theorem 2 was to derive conditions under which $\phi'_{FP} \geq 1$ holds everywhere. Knowing when this is true has at least one other interesting implication.

**Proposition 9** Assume that $F_1 = F_2$. If $\phi' \geq 1$ everywhere, then $\beta_1$ and $\beta_2$ lie on either side of the symmetric equilibrium strategy $\beta_s$ of a standard first price auction

$$\beta_1 \geq \beta_s \geq \beta_2.$$  

This is intuitive: 1 bids less aggressively than if he were not favored, and 2 more. Overall, costs go up, but, for appropriately chosen $A$, the improvement in efficiency more than compensates the buyer. Fig. 5 shows how $\beta_1$ and $\beta_2$ vary in $A$ for the uniform case. It is an interesting conjecture that $\beta_1$ and $\beta_2$ should move monotonically further apart as $A$ grows for general $f$.

### 8 Conclusion

It is interesting to see what else the concept of $\rho$-concavity can do for us, either in addressing other auction-related questions, and in mechanism design more generally. The fact that the slope of a bid function is precisely one minus the local $\rho$-concavity of the associated surplus function, and that this in turn is tightly tied to the properties of the underlying distributions certainly suggests that the connection is worth a fair amount of exploration. It would,
in specific, be interesting to see what our tools have to say about first-price auctions with more than 2 players. We are also hopeful that they will be useful in understanding the effects of auction design on ex-ante incentives for firms to invest in quality or in improving their cost distribution.

In the specific context of understanding the ranking of procurement auctions, these tools took us quite far. For a good sized class of auctions, we have provided theoretical foundations for the superiority of second-price mechanisms over first price mechanisms. These results should be interesting to an economic theorist, but also to firms that engage in procurement.

Our derivation of the result that $\phi_{FP} > 1$ uses techniques that are new, and that seem likely have more generally applicability. In particular, the degree to which one can generate bounds on the surplus available to each player, and use that to partially characterize equilibrium bid functions seems intriguing. An obvious topic for further research is to get a better understanding of the examples suggesting that $\phi_{FP} > 1$ holds more widely than under our conditions. The key step there seems to be to develop tighter bounds on the surplus in asymmetric auctions.

Our ranking results are primarily in terms of dominance. Further exploration of the effects of considering expected buyer surplus instead seems merited. Other simple auction forms, such as percentage-bonus auctions,
deserve more consideration, especially because of their wide-spread use in practice.\textsuperscript{45}

9 Appendix

9.1 Proofs for Section 4

An immediate consequence of Proposition 1 is the following lemma. See Karlin (1968) for a different proof.

**Lemma 21** If $f$ is a log-concave density then $F$ and $\bar{F}$ are themselves log-concave. If $f$ is a (strictly) log-concave density then $\frac{f}{F}$ is (strictly) decreasing and $\frac{1}{F}$ is (strictly) increasing.

**Lemma 22** Let $g$ be log-concave on some interval near 1, with $g(1) = 0$. Then,

$$W_G(1) = \frac{1}{2 - W_g(1)}.$$  

**Proof** By assumption $g(1) = 0$. Let $c^*$ be such that $g$ is log-concave on $[c^*, 1]$ and such that $g'(c^*) < 0$. Then, by log-concavity, $g'(s) < 0$ for all $s \in [c^*, 1)$. Thus, l’Hôpital’s rule applies to give

$$W_G(1) = \lim_{s \to 1} \left( \frac{-g'(s)\bar{G}(s)}{g''(s)} \right) = -\lim_{s \to 1} \frac{g''(s)\bar{G}(s) - g'(s)g(s)}{2g'(s)g(s)} = -\lim_{s \to 1} \frac{g''(s)\bar{G}(s)}{2g'(s)g(s)} + \frac{1}{2} = \frac{1}{2} \lim_{s \to 1} \left( \frac{g''(s)g(s)g'(s)\bar{G}(s)}{(g')^2(s)g^2(s)} \right) + \frac{1}{2} = \frac{1}{2} \left( W_G(s)W_g(s) + \frac{1}{2} \right) = \frac{1}{2} \left( W_G(1)W_g(1) + 1 \right),$$

where the last step is valid noting that by Proposition 1 $\bar{G}$ is log-concave on $[c^*, 1]$ and that since $g$ is decreasing near zero, $W_G(1) \geq 0$, and so $W_G(1)$ is finite. Rearranging yields the result. \hfill \blacksquare

In the main text, we contended that it is mild to assume that $W_h(1)$ is finite when $h(1) = 0$, and very mild to assume that $W_H$ was finite. The following lemma shows why.

\textsuperscript{45}See Marion (2006) and (2007).
Lemma 23 Assume that $g$ is $C^\infty [0, 1]$ and has $g(1) = 0$ and $g^{(k)}(1) < \infty$ for all $k$. Then, $W_g(1)$ is finite.

Proof Let $n$ be such that $g^{(n)}(1) \neq 0$ while $g^{(k)}(1) = 0$ for all $k < n$. Note that $n$ must be finite\textsuperscript{46}. Since $g^{(n)}(1) \neq 0$ while $g^{(n-1)}(1) = 0$.

\[ W_{g^{(n-1)}}(1) \equiv \frac{g^{(n-1)}(1)g^{(n+1)}(1)}{(g^{(n)}(1))^2} = 0. \textsuperscript{47}

Assume by induction that $W_{g^{(n-k)}}(1) = \frac{k-1}{k}$ for some $k \in \{1, 2, \cdots, n\}$. Then, since $g^{n-k}(1) = 0$, Lemma 22 applies to $g^{(n-k)}$ (which, since $W_{g^{(n-k)}}(1) < 1$, is log-concave on some interval near 1) to yield

\[ W_{g^{(n-(k+1))}}(1) = \frac{1}{2 - W_{g^{(n-1)}}(1)} = \frac{1}{2 - \frac{k-1}{k}} = \frac{k}{2k - (k - 1)} = \frac{k}{k+1}. \]

So, when $h(1) = 0$, or, noting that $\bar{H}(1) = 0$ by definition, all that is required is enough differentiability that the first non-zero derivative is not swamped by the next derivative up.

Proof of Lemma 1 Make the substitution $\rho_h(c) = 1 - W_h(c)$, and manipulate, noting that since $h$ is log-concave, $W_h(c) \leq \bar{W}_h(c) \leq 1$, while $W_{\bar{H}} > 0$ when $h' < 0$, and so the cross multiplications are valid. \hfill \blacksquare

Proof of Lemma 2 Since $W_h$ is continuous and $W_h(1)$ is finite,

\[ \underline{W}_h(1) = W_h(1) = \bar{W}_h(1). \]

The result is then immediate from Lemma 22 applied to $g = h$. \hfill \blacksquare

\textsuperscript{46} Otherwise by Taylor’s formula $f \equiv 0$.

\textsuperscript{47} The sole use of the assumption that $g \in C^\infty [0, 1]$ is to assure that $\frac{g^{(n+1)}(1)}{(g^{(n)}(1))^2}$ is finite. Clearly substantially weaker conditions than $h \in C^\infty [0, 1]$ would suffice.
9.2 Proofs for Section 5

Proof of Theorem 2 We proceed in a sequence of steps.

Step 1: Derivation of (6) and (7). If 1 with type \( c \) bids as if his type is \( \tilde{c} \), his surplus is \( S_1(\tilde{c}; c) = \tilde{F}(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c) \). By the envelope theorem,

\[
\frac{\partial}{\partial c} S_1(c) = \frac{\partial}{\partial c} \tilde{S}_1(\tilde{c}; c) \bigg|_{\tilde{c}=c} = \tilde{F}(\psi(c)).
\]

Given \( b_1 \leq \tilde{c}_1, S_1(\tilde{c}_1) = 0 \), yielding (6). Similarly, \( \frac{\partial}{\partial c} S_1(c) = \tilde{F}(\phi(c)) \), and for \( c_2 > \tilde{c}_1 - A \) no \( b_2 > c_2 \) ever wins, and so \( S_2(\tilde{c}_1 - A) = 0 \), yielding (7).

Step 2: Derivation of (8) and (9). From (6) we have

\[
\tilde{F}(\psi(c))(\beta_1(c) - c) = S_1(c) = \int_c^{\tilde{c}_1} \tilde{F}(\psi(s))ds,
\]

and (8) follows by rearranging, and similarly for (9).

Step 3: Positive derivatives of \( \beta_1, \beta_2 \) and \( \phi' \). As increasing functions, \( \beta_1(\cdot), \beta_2(\cdot), \phi(\cdot), \) and \( \psi(\cdot) \) are differentiable almost everywhere. Note that a bid of \( b_1 \in [\beta_2(0) + A, \tilde{c}_1) \) by 1 wins with probability \( \tilde{F} \left( \beta_2^{-1} \left( \tilde{b}_1 - A \right) \right) \). Pick \( \tilde{b}_1 \) such that \( \beta_2 \) is differentiable at \( \beta_2^{-1} \left( \tilde{b}_1 - A \right) \), let \( c_M = \beta_2^{-1} \left( \tilde{b}_1 - A \right) \), and let \( \tilde{c}_1 = \beta_1^{-1} \left( \tilde{b}_1 \right) \). Then, since

\[
\tilde{F}(\beta_2^{-1}(b_1 - A))(b_1 - \tilde{c}_1)
\]

is maximized at \( \tilde{b}_1 \),

\[
-f \left( \beta_2^{-1}(\tilde{b}_1 - A) \right) \frac{1}{\beta_2' \left( \beta_2^{-1}(\tilde{b}_1 - A) \right)} \left( \tilde{b}_1 - \tilde{c}_1 \right) + \tilde{F} \left( \beta_2^{-1}(\tilde{b}_1 - A) \right) = 0.48
\]

(31)

Since \((b_1 - \tilde{c}_1) > 0\) (because bidder 1 earns positive surplus with \( \tilde{c}_1 \leq c_1 \), and \( \tilde{c}_1 < c_1 \) since \( \tilde{b}_1 < c_1 \)), and since the other terms in (31) are finite but positive,

\[
\frac{1}{\beta_2' \left( \beta_2^{-1}(\tilde{b}_1 - A) \right)}
\]

48 At \( \tilde{b}_1 = \beta_2(0) + A \), this is weakly negative.
is finite and positive as well, and so $\beta_2' \left( \beta_2^{-1} \left( \hat{b}_1 - A \right) \right) > 0$. Hence, wherever $\beta_2'$ exists, $\beta_2' > 0$. Similarly, for $c_2 < \hat{c}_1 - A$, if $\beta_1'$ exists at $\phi(c_2)$, then $\beta_1' > 0$. As $\beta_1(\phi(c)) = \beta_2(c) + A$, where $\beta_1'$ and $\beta_2'$ exist,

$$\phi'(c) = \frac{\beta_2'(c)}{\beta_1'(\phi(c))} > 0.$$  \hspace{1cm} (32)

**Step 4: Derivation of (10) and (11).** Using (8)

$$\begin{align*}
\beta_1'(c) &= 1 + \left( \int_c^{\epsilon_1} \frac{\bar{F}_2(\psi(s))ds}{\bar{F}_2(\psi(c))} \right)' \\
&= 1 + \left( \int_c^{\epsilon_1} \frac{\bar{F}_2(\psi(s))ds}{\bar{F}_2(\psi(c))} \right)' - \left( \int_c^{\epsilon_1} \frac{\bar{F}_2(\psi(s))ds}{\bar{F}_2(\psi(c))} \right)' \\
&= 1 + \left( \int_c^{\epsilon_1} \frac{\bar{F}_2(\psi(s))(-\bar{F}_2(\psi(c))) + \int_c^{\epsilon_1} \bar{F}_2(\psi(s))ds f_2(\psi(c))\psi'(c)}{(\bar{F}_2(\psi(c)))^2} \\
&= \int_c^{\epsilon_1} \frac{\bar{F}_2(\psi(s))ds f_2(\psi(c))}{\bar{F}_2(\psi(c))} \frac{1}{\phi'(\psi(c))},
\end{align*}$$

using that $\phi' > 0$ wherever $\beta_1'$ and $\beta_2'$ exist. Similarly,

$$\begin{align*}
\beta_2'(c) &= \int_c^1 \bar{F}_1(\phi(s))ds \frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} \phi'(c) \\
&= S_1(c) \frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} \phi'(c).
\end{align*}$$

**Step 5: Derivation of (12).** Substituting (10) and (11) into (32) gives

$$\begin{align*}
\phi'(c) &= \frac{S_2(c) \frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} \phi'(c)}{S_1(\phi(c)) \frac{f_2(c)}{\bar{F}_2(c) \phi'(\phi(c))}} \\
&= \frac{S_2(c) \frac{f_1(\phi(c))}{\bar{F}_1(\phi(c))} \phi'(c)}{S_1(\phi(c)) \frac{f_2(c)}{\bar{F}_2(c) \phi'(\phi(c))}}.
\end{align*}$$

Canceling $\phi'(c) > 0$ and rearranging yields (12).
Step 6: Continuous differentiability of $\phi$. By (12),

$$
\phi'(c) = \frac{S_1(\phi(c))}{S_2(c)} \frac{f_2(c)}{F_2(c)}
$$

almost everywhere. As a bounded, continuous function on a compact interval, $\phi$ is absolutely continuous (see, e.g., Wade (1995)) and so (see, e.g., Billingsley, Theorem 31.8).

$$
\phi(c) = \phi(0) + \int_0^c \phi'(t) \, dt = \int_0^c \frac{S_1(\phi(t))}{S_2(t)} \frac{f_2(c)}{F_2(c)} \, dt.
$$

Since $\phi$ is continuous, $S_1(\phi(c)) \frac{f_2(c)}{F_2(c)} \frac{F_2(c)}{F_2'(\phi(c))}$ is arbitrary, the result follows. Pick $a < \bar{c}_1 - A$.

Step 7: $\phi \in C^{k+1}[0, \bar{c}_1 - A]$. We show the result on $[0, a]$, $a < \bar{c}_1 - A$. Since $a$ is arbitrary, the result follows. Pick $a < \bar{c}_1 - A$. From Step 6, $\phi(c)$ is $C^1$ on $[0, \bar{c}_1 - A]$, with

$$
\phi'(c) = \frac{\int_{\phi(c)}^{c} \bar{F}_2(\psi(s)) \, ds}{\int_{c}^{\bar{c}_1 - A} \bar{F}_1(\phi(c))} \frac{f_2(c)}{F_2(c)} \frac{F_2(c)}{F_2'(\phi(c))}.
$$

(33)

For $1 \leq \bar{k} \leq k$, assume that $\phi \in C^{\bar{k}}[0, a]$, Note that,

$$
\left( \int_{\phi(c)}^{c} \bar{F}_1(\psi(s)) \, ds \right)' = -\phi'(c) \bar{F}_1(c).
$$

Since $\phi' \in C^{\bar{k}-1}[0, a]$ and $\bar{F}_1 \in C^k[0, \bar{c}_1]$, $-\phi'(c) \bar{F}_1(c) \in C^{\bar{k}-1}[0, a]$, and so $\int_{\phi(c)}^{c} \bar{F}_1(\psi(s)) \, ds \in C^k[0, a]$. Similarly, $\left( \int_{c}^{\bar{c}_1 - A} \bar{F}_2(\phi(c)) \right)' = -\bar{F}_2(\phi(c))$, and so $\int_{c}^{\bar{c}_1 - A} \bar{F}_2(\phi(c)) \in C^{\bar{k}+1}[0, a]$. Since $\phi$, $\bar{F}$ each belong to $C^k[0, a]$, and since all components of the RHS of (33) are everywhere positive on $[0, a]$, the RHS of (33) belongs to $C^{\bar{k}}[0, a]$ (see Shilov (1997). Thus, $\phi' \in C^{\bar{k}}[0, a]$, and so $\phi \in C^{\bar{k}+1}[0, a]$. By induction, $\phi \in C^{k+1}[0, a]$.

Step 8: $\beta_1 \in C^{k+1}[0, \bar{c}_1]$ and $\beta_2 \in C^{k+1}[0, \bar{c}_1 - A]$: This follows immediately using the argument and conclusion of Step 7 applied to (10) and (11).
Proof of Lemma 4 Assume $W_{F \circ \gamma}$ is increasing and $\gamma$ is concave. Then,

$$W_{F \circ \gamma}(c) = \frac{\gamma'(c) F(\gamma(c)) \int_c^1 \tilde{F}(\gamma(s)) ds}{(F(\gamma(c)))^2}$$

and so if

$$\gamma'(c) \int_c^1 \tilde{F}(\gamma(s)) ds \geq \int_{\gamma(c)}^1 \tilde{F}(s) ds$$

then

$$W_{F \circ \gamma}(c) \geq W_{F}(\gamma(c)) \geq W_{F}(c)$$

since $W_{F}$ is increasing and $\gamma(c) \geq c$.

To prove 34 define

$$h(c) = \gamma'(c) \int_c^1 \tilde{F}(\gamma(s)) ds - \int_{\gamma(c)}^1 \tilde{F}(s) ds.$$ 

Since $\gamma$ is concave, $\lim_{c \to 1} \gamma'(c) < \infty$, and so $h(1) = 0$. Also,

$$h'(c) = \gamma''(c) \int_c^1 \tilde{F}(\gamma(s)) ds - \gamma'(c) \tilde{F}(\gamma(c)) + \gamma'(c) \tilde{F}(\gamma(c))$$

$$= \gamma''(c) \int_c^1 \tilde{F}(\gamma(s)) ds \leq 0$$

again since $\gamma$ is concave. Since $h$ is decreasing and $h(1) = 0$ we thus have $h(c) \geq 0$ or equivalently (34). The other case follows *mutatis mutandi.*

Proof of Lemma 5 Define

$$Q(c) = \alpha \int_c^1 h^\alpha(s) ds - \alpha - 1 \int_c^1 h(s) ds.$$ 

Note that $Q(1) = 0$ and that

$$Q'(c) = -\alpha h^\alpha(c) + h^\alpha(c) - (\alpha - 1) h'(c) h^{\alpha-2}(c) \int_c^1 h(s) ds$$

$$= (1 - \alpha) h^\alpha(c) \left(1 + \frac{h'(c) \int_c^1 h(s) ds}{h^2(c)}\right)$$

$$= (1 - \alpha) h^\alpha(c) \rho_f h(c)$$

$$\leq 0$$
since by assumption \( \rho_h \geq 0 \) so by Proposition 1 \( \rho_{fh} (c) \geq 0 \). Thus \( Q' (c) \leq 0 \) and \( Q(c) \geq 0 \).

But

\[
W_{fh^\alpha} (c) = -\frac{\alpha h' (c) h^{\alpha - 1} \int_c^1 h^\alpha (s) ds}{h^{2\alpha} (c)} = -\frac{h' (c) \int_c^1 h^\alpha (s) ds}{h^{\alpha + 1}}
\]

and

\[
W_{fh} (c) = -\frac{h' (c) \int_c^1 h(s) ds}{h^2}
\]

and so

\[
W_{fh^\alpha} (c) - W_{fh} (c) = \frac{-h' (c)}{h^{\alpha + 1} (c)} \left( \alpha \int_c^1 h^\alpha (s) ds - h^{\alpha - 1} (c) \int_c^1 h(s) ds \right) = \epsilon Q(c) \geq 0
\]

since \( h' \leq 0 \). 

**9.3 Proofs for Section 6**

**Proof of Lemma 6** Since \( \bar{F}_1 (\gamma_F (c)) = \bar{F}_2 (c) \),

\[
f_1 (\gamma (c)) \gamma'_{F} (c) = f_2 (c),
\]

and so

\[
\frac{f_2}{F_2} (c) = \gamma'_{F} (c) \frac{f_1}{F_1} (\gamma_F (c)),
\]

and so, since \( \gamma'_{F} \geq 1 \),

\[
\frac{f_1}{F_1} (\gamma_F (c)) - \frac{f_2}{F_2} (c) = (1 - \gamma'_{F} (c)) \frac{f_1}{F_1} (\gamma_F (c)) \leq 0.
\]

Thus as \( \frac{f_1}{F_1} \) is increasing, \( \gamma'_{F} \geq \gamma_F \), and in particular, \( \gamma_F (0) \leq \gamma'_{F} (0) \).

Similarly,

\[
f_1 (\gamma (c)) \gamma''_{F} (c) + f_1' (\gamma (c)) (\gamma'_{F} (c))^2 = f_2' (c),
\]

and so

\[
W_{F_2} (c) = \bar{F}_2 (c) = \frac{\bar{F}_1 (\gamma_F (c)) \left( -f_1 (\gamma (c)) \gamma''_{F} (c) - f_1' (\gamma (c)) (\gamma'_{F} (c))^2 \right)}{(f_1 (\gamma_F (c)) \gamma'_{F} (c))^2}
\]

\[
= \frac{\bar{F}_1 (\gamma_F (c)) f_1 (\gamma (c)) \gamma''_{F} (c)}{(f_1 (\gamma_F (c)) \gamma'_{F} (c))^2}
\]

\[
\leq W_{F_1} (\gamma_F (c)).
\]
But then, since $W_{F_1}$ is increasing, and since $\gamma_{F_1}(c) \geq \gamma_F(c)$,

$$W_{F_1} \left( \gamma_{F_1}(c) \right) \geq W_{F_2}(c). \quad (35)$$

Since $\frac{\tilde{F}_1}{\tilde{f}_1} \left( \gamma_{F_1}(c) \right) = \frac{\tilde{F}_2}{\tilde{f}_2}(c) > 0$,

$$-\frac{\tilde{f}_1'}{\tilde{f}_1} \left( \gamma_{F_1}(c) \right) \geq -\frac{\tilde{f}_2}{\tilde{f}_2}(c)$$

and so $\gamma_{F_1} \leq \gamma_{F_2}$. Finally, differentiating $\frac{\tilde{F}_1}{\tilde{f}_1} \left( \gamma_{F_1}(c_2) \right) = \frac{\tilde{F}_2}{\tilde{f}_2}(c_2)$, noting that $W_{F_1} \leq 1$ by log-concavity, and using (35)

$$\gamma_{F_1}'(c) = \frac{-1 + W_{F_2}(c)}{-1 + W_{F_1} \left( \gamma_{F_1}(c) \right)} \geq 1. \quad \blacksquare$$

**Proof of Lemma 7** Consider an incentive compatible mechanism $\mu$ in which the buyer always buys. Adapting Myerson (1981) in the obvious ways to the setting, $BS(\phi_\mu)$ is

$$\int \int \left( I_{\{c_1 < \phi_\mu(c_2)\}} \left( v_1 - \omega_1(c_1) \right) + (1 - I_{\{c_1 < \phi_\mu(c_2)\}}) \left( v_2 - \omega_2(c_2) \right) \right) f_1(c_1) f_2(c_2) dc_1 dc_2$$

$$= \int \int (v_2 - \omega_2(c_2)) f_1(c_1) f_2(c_2) dc_1 dc_2 \quad \text{Term 1}$$

$$+ \int \int I_{\{c_1 < \phi_\mu(c_2)\}} (v_1 - v_2 - (\omega_1(c_1) - \omega_2(c_2))) f_1(c_1) f_2(c_2) dc_1 dc_2. \quad \text{Term 2}$$

The lemma follows since (recalling $\omega_2(c_2) = c_2 + \frac{F_2}{f_2}(c_2)$, and integrating out $c_1$), Term 1 equals

$$\int \left( v_2 - c_2 - \frac{F_2}{f_2}(c_2) \right) f_2(c_2) dc_2 = v_2 - E(c_2) - \int F_2(c_2) dc_2$$

$$= v_2 - E(c_2) - 1 + \int (1 - F_2(c_2)) dc_2$$

$$= v_2 - E(c_2) - 1 + E(c_2)$$

$$= v_2 - 1. \quad \blacksquare$$
Proof of Lemma 8

Differentiating (18) we have

$$\phi'_M(c_2) = \frac{1 + \rho_{F_2}(c_2)}{1 + \rho_{F_1}(\phi_M(c_2))}. \quad (36)$$

By A1, $\gamma'_F(c_2) \geq 1$ and $\gamma''_F(c_2) \geq 0$. Since $F_1(\gamma_F(c_2)) = F_2(c_2)$

$$\gamma'_F(c_2)f_1(\gamma_F(c_2)) = f_2(c_2)$$

$$\left(\gamma'_F(c_2)\right)^2 f'_1(\gamma_F(c_2)) + \gamma''_F(c_2)f_1(\gamma_F(c_2)) = f'_2(c_2)$$

and

$$W_{F_1}(\gamma_F(c_2)) + \frac{\gamma''_F(c_2)}{\left(\gamma'_F(c_2)\right)^2} \left(\frac{F_1(\gamma_F(c_2))}{f_1(\gamma_F(c_2))}\right) = W_{F_2}(c_2)$$

and so

$$W_{F_1}(\gamma_F(c_2)) \leq W_{F_2}(c_2).$$

By A3 since $W_{F_1}$ is increasing, for any $c_1 \geq \gamma_F(c_2)$

$$\rho_{F_1}(c_1) \geq \rho_{F_2}(c_2).$$

In particular at a point where $c_1 = \gamma_F(c_2)$ we have by (36) and A1 that

$$\phi'_M(c_2) < 1 \leq \gamma'_F(c_2). \quad \blacksquare$$

Proof of Lemma 9

Note that

$$f = -\bar{F}' = n\bar{H}^{n-1}h,$$

and so

$$\frac{f}{\bar{F}} = \frac{n}{H} \quad (37)$$

and

$$\frac{f'}{f} = -(n-1)\frac{h}{H} + \frac{h'}{h}. \quad (38)$$

Thus,

$$W_{\bar{F}} = \frac{\bar{F} - f'}{f}$$

$$= \frac{1}{n} \frac{\bar{H}}{h} \left( (n-1)\frac{h}{H} - \frac{h'}{h} \right)$$

$$= \frac{n-1}{n} \frac{1}{h} \frac{\bar{H}h'}{n}$$

$$= \frac{n-1}{n} + \frac{1}{n} W_{\bar{H}}.$$

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So, \( W_F \) has the same monotonicity as \( W_H \).

Using (37) and (38), note that

\[
\frac{(\log f)''}{(\log F)''} = \frac{\left(\frac{-h'}{n}\right)'}{\left(\frac{f}{F}\right)'} = \frac{(n-1)\left(\frac{h'}{n}\right)'}{n \left(\frac{h}{n}\right)'}
\]

\[
= \frac{(n-1)}{n} + \frac{1}{n} \left(\frac{-h'}{n}\right)'
\]

\[
= \frac{(n-1)}{n} + \frac{1}{n} (\log h)''.
\]

So, if \( \frac{(\log h)''}{(\log H)''} \) reaches its minimum at \( c \), so does \( \frac{(\log f)''}{(\log F)''} \).

**Proof of Lemma 10** By (18)

\[
\Delta + c_2 + \frac{F_2}{f_2} (c_2) = \phi_M (c_2) + \frac{F_2}{f_2} (\phi_M (c_2))
\]

but if \( f_1 (\tilde{c}_1) = f_2 (1) = 0 \) the left-hand side diverges as \( c_2 \) approaches 1 and thus \( \phi_M (1) = \tilde{c}_1 \). Imagine there exists a \( \tilde{c} \in (0,1) \) such that

\[
\phi_M (c) > \gamma_F (c)
\]

for all \( c \in (\tilde{c},1) \). Thus for all \( c \in (\tilde{c},1) \)

\[
\Delta > \gamma_F (c) + \frac{F_1 (\gamma_F (c))}{f_1 (\gamma_F (c))} - \left( c + \frac{F_2 (c)}{f_2 (c)} \right)
\]

(39)

but by A1, \( F_1 (\gamma_F (c)) = f_2 (c) \) and \( \gamma_F' (c) f_1 (\gamma_F (c)) = f_2 (c) \). So (39) becomes

\[
\Delta > \gamma_F (c) + \frac{F_1 (\gamma_F (c))}{f_1 (\gamma_F (c))} - \left( 1 + \frac{F_2 (c)}{f_2 (c)} \right)
\]

\[
= \frac{\gamma_F (c) - c}{f_1 (\gamma_F (c))} \left( 1 - \frac{1}{\gamma_F' (c)} \right).
\]

But, since \( \gamma_F' (0) \geq 1 \), \( \gamma_F' \) is monotone and \( \gamma_F \) is not the identity, \( \gamma_F' (c) > 1 \) near 1, and so the RHS expression diverges, a contradiction.

Thus there exists some \( \hat{c} \) close to 1 for which \( \phi_M (\hat{c}) < \gamma_F (\hat{c}) \). Since \( \gamma_F' (c) \geq 1 \), and since \( \phi_M (1) = \gamma_F (1) \), it must be that for some \( c^* \in (\hat{c},1) \), \( \phi_M (c^*) > 1 \).
9.4 Proofs for Subsection 6.8

Proof of Lemma 13 From Lemma 6 $\phi_{FP}(0) = \gamma_F(0) \leq \gamma_F'(0)$. So, since $\gamma_F'(c_2) \geq 1$ and $\gamma_A'(c_2) = 1$ for all $c_2$, if $\phi_{FP}(c_2) > \max \left( \gamma_F'(c_2), \lambda_A(c_2) \right)$ anywhere, then there is $c_M$ where $\phi_{FP}'(c_2) > 1$, and where $\phi_{FP}(c_2) \geq \max \left( \gamma_F'(c_2), \lambda_A(c_2) \right)$. This contradicts Lemma 11. So, $\phi_{FP} \leq \max \left( \gamma_F', \lambda_A \right) < \lambda_T$. ■

Proof of Lemma 16 We prove this through a series of lemmas.\textsuperscript{49}

Lemma 24 For all $f$ log-concave with support $[0, 1]$ we have

$$\limsup_{s \to 1} \left( \frac{\tilde{F}(s)}{f(s)} \right)' \in [-1, 0].$$

Proof of Lemma 24 Since $\left( \frac{\tilde{F}(s)}{f(s)} \right)' = -1 + W_F(s) = -\rho(s)$, $\left( \frac{\tilde{F}(s)}{f(s)} \right)' \leq 0$ by log-concavity. Since $f$ is log-concave, $f$ is uni-modal, and hence monotone on $[c, 1]$ for some $c$. Assume first that $f$ is non-decreasing on $[c, 1]$. Then, $\lim_{s \to 1} W_F(s) = 0$, since $\frac{\tilde{F}}{f}$ is non-increasing by log-concavity and so $\lim_{s \to 1} \left( \frac{\tilde{F}(s)}{f(s)} \right)$ is well defined, finite, and positive, and since $\lim_{s \to 1} \left( \frac{\tilde{F}(s)}{f(s)} \right) = 0$.

Assume $f$ is decreasing on $[c, 1]$ (this includes the case $f(1) = 0$). Then, for $s \in (c, 1]$, $W_F(s) \geq 0$, and so $\liminf_{s \to 1} (-1 + W_F(s)) \geq -1$. ■

Lemma 25 $\beta_1$ is differentiable at $\bar{c}_1$ with $\beta'_1(\bar{c}_1) = 0$. $\beta_2$ is continuously differentiable at $\bar{c}_1 - A$ with $\beta'_2(\bar{c}_1 - A) = 0$.

For $\beta_1$, we claim differentiability at $\bar{c}_1$, but not continuous differentiability.\textsuperscript{50} For $\beta_2$, we assert continuous differentiability. The proofs of the two cases are quite different: with $c_1 = \bar{c}_1$, 1 already has probability $\tilde{F}(\bar{c}_1 - A)$ of winning while with $c_2 = \bar{c}_1 - A$, 2 has no chance of winning and so $S_1$ and $S_2$ have different shapes at the top. Thus, even bidding $\bar{c}_1$ already earns 1 a great deal of surplus, and this bounds his bid from below sufficiently to guarantee $\beta'_1(\bar{c}_1) = 0$. Showing that $\beta'_2(c_2) \rightarrow 0$ as $c_2 \rightarrow \bar{c}_1 - A$ involves an examination of 1’s incentives.

\textsuperscript{49}The proof is more straightforward if $\lim \varphi'$ is known to exist.

\textsuperscript{50}If $h$ is continuously differentiable on $[0, 1)$, its derivative at 1 need not equal $\lim_{s \to 1} h'(x)$ (which need not exist). We establish continuous differentiability of $\beta_1$ later in a rather round-about way.
Proof of Lemma 25 Since $\beta_1$ is increasing,
\[ \liminf \frac{\beta_1(c_1) - \beta_1(c)}{c_1 - c} \geq 0. \]
Assume
\[ \limsup \frac{\beta_1(c_1) - \beta_1(c)}{c_1 - c} = \alpha > 0. \tag{40} \]
For any $c$, $\beta_1(c)$ earns $F_2(\psi(c)) (\beta_1(c) - c)$, while a bid of $c_1$ earns $F_2(c_1 - A)(c_1 - c)$. Since $\beta_1(c)$ is a best response,
\[ F_2(\psi(c)) (\beta_1(c) - c) \geq F_2(c_1 - A)(c_1 - c), \]
and so for $c < c_1$,
\[ \frac{\beta_1(c) - c}{c_1 - c} \geq \frac{F_2(c_1 - A)}{F_2(\psi(c))}. \]
But, as $c \rightarrow c_1$, $\psi(c) \rightarrow c_1 - A$, and so
\[ \liminf \frac{\beta_1(c) - c}{c_1 - c} \geq 1. \tag{41} \]
By (40), along a subsequence $c_t$, $c_t \rightarrow c_1 - A$,
\[ \frac{\beta_1(c_1) - \beta_1(c_t)}{c_1 - c_t} \geq \frac{\alpha}{2}. \]
But, since
\[ \frac{\beta_1(c_1) - \beta_1(c_t)}{c_1 - c_t} + \frac{\beta_1(c_t) - c_t}{c_1 - c_t} = 1, \]
this means that
\[ \frac{\beta_1(c_t) - c_t}{c_1 - c_t} < 1 - \frac{\alpha}{2} \]
for all $t$, contradicting (41). So,
\[ \liminf \frac{\beta_1(c_1) - \beta_1(c)}{c_1 - c} = \limsup \frac{\beta_1(c_1) - \beta_1(c)}{c_1 - c} = 0, \]
and $\beta_1$ is differentiable at $c_1$, with $\beta_1'(c_1) = 0$.

Player 1’s profit from mimicking $\tilde{c}$ is $F_2(\psi(\tilde{c}))(\beta_1(\tilde{c}) - c)$. Taking the first order condition at $\tilde{c} = c$,
\[ F_2(\psi(c)) \beta_1'(c) = f_2(\psi(c)) \psi'(c) (\beta_1(c) - c). \]
But,
\[ \psi'(c) = \frac{\beta_1'(c)}{\beta_2(\psi(c))}. \]
and so

\[ \tilde{F}_2(\psi(c)) \beta_1'(c) = f_2(\psi(c)) \frac{\beta_1'(c)}{\beta_2'(\psi(c))} (\beta_1(c) - c). \]

Cancelling \( \beta_1'(c) > 0 \), and rearranging,

\[ \beta_2'(\psi(c)) = \frac{f_2(\psi(c))}{\tilde{F}_2(\psi(c))} (\beta_1(c) - c) < \frac{f_2(\bar{c}_1 - A)}{\tilde{F}_2(\bar{c}_1 - A)} (\bar{c}_1 - c). \]

Thus, as \( c \to \bar{c}_1 \), \( \beta_2'(\psi(c)) \to 0. \) But then, \( \beta_2'(\bar{c}_1 - A) \) exists and equals 0.

\[ \blacksquare \]

An easy but useful implication of this is:

**Lemma 26** As \( c \to \bar{c}_1 - A \),

\[ \frac{\beta_1(\phi(c)) - \phi(c)}{\bar{c}_1 - \phi(c)} \to 1 \text{ and } \frac{\beta_2(c) - c}{\bar{c}_1 - A - c} \to 1. \]

**Proof** From Lemma 25, we have \( \beta_1'(\bar{c}_1) = 0 \), and hence

\[ \frac{\bar{c}_1 - \beta_1(\phi(c))}{\bar{c}_1 - \phi(c)} \to 0 \]

as \( c \to \bar{c}_1 - A \). But then, since for all \( c \)

\[ \frac{\bar{c}_1 - \beta_1(\phi(c))}{\bar{c}_1 - \phi(c)} + \frac{\beta_1(\phi(c)) - \phi(c)}{\bar{c}_1 - \phi(c)} = 1, \]

\[ \frac{\beta_1(\phi(c)) - \phi(c)}{\bar{c}_1 - \phi(c)} \to 1. \]

The proof for \( \beta_2 \) is identical. \( \blacksquare \)

**Lemma 27** \( \lim \sup \phi'(c) \in \{0, \infty\} \).

**Proof** Assume that \( \lim \sup \phi'(c) = \alpha \in (0, \infty) \). Now,
\[
\phi'(c) = \frac{S_1(\phi(c)) f_2(c) F_2^2(c)}{S_2(c) f_1(\phi(c)) F_1^2(c)} = \frac{(\beta_1(\phi(c)) - \phi(c)) f_2(c)}{F_2^2(c)} = \frac{(\beta_1(\phi(c)) - \phi(c)) F_1(\phi(c))}{f_1(\phi(c)) F_1^2(c)} = \frac{(\beta_1(\phi(c)) - \phi(c)) F_1(\phi(c))}{f_1(\phi(c)) F_1^2(c)} = \frac{\beta_1(\phi(c)) - \phi(c)}{\bar{c}_1 - \phi(c)} \frac{F_1(\phi(c))}{f_1(\phi(c))}.
\]

But then, by Lemma 26

\[
\limsup \phi'(c) = \lim \frac{\beta_1(\phi(c)) - \phi(c)}{\bar{c}_1 - \phi(c)} \limsup \frac{\bar{c}_1 - \phi(c) F_1(\phi(c))}{f_1(\phi(c)) F_2^2(c)} = \lim \frac{F_1(\phi(c))}{f_1(\phi(c)) F_2^2(c)} = \frac{f_2}{F_2} \left( \frac{\bar{c}_1 - A}{\bar{c}_1 - A - c} \right) \limsup \frac{\bar{c}_1 - \phi(c) F_1(\phi(c))}{(\bar{c}_1 - A - c)}.
\]

Since the top and bottom of

\[
\frac{\bar{c}_1 - \phi(c) F_1(\phi(c))}{(\bar{c}_1 - A - c)}
\]
go to 0 as \( c \to \bar{c}_1 - A \), a generalization of l’Hôpital’s rule (Lee (1977)) gives

\[
\limsup \phi'(c) \leq \frac{f_2(\bar{c}_1 - A) \limsup \frac{\partial}{\partial c} \left( \frac{(\bar{c}_1 - \phi(c)) F_1'(\phi(c))}{f_1(\phi(c))} \right)}{F_2(\bar{c}_1 - A)} \limsup \frac{\partial}{\partial c} ((\bar{c}_1 - A - c))
\]

\[
= \frac{f_2(\bar{c}_1 - A)}{F_2(\bar{c}_1 - A)} \limsup \frac{-\phi'(c) F_1'(\phi(c)) + (\bar{c}_1 - \phi(c)) \left( \frac{F_1'(s)}{f_1(s)} \right)_{s=\phi(c)}' \phi'(c)}{-1}
\]

\[
= \frac{f_2(\bar{c}_1 - A)}{F_2(\bar{c}_1 - A)} \left( \limsup \phi'(c) \left( \frac{\tilde{F}_1'(\phi(c)) + (\bar{c}_1 - \phi(c)) \left( \frac{-\tilde{F}_1'(s)}{f_1(s)} \right)_{s=\phi(c)}'}{f_1'(\phi(c))} \right) \right)
\]

Since \( \limsup \phi'(c) \in (0, \infty) \) by assumption, cancel to obtain

\[
1 \leq \frac{f_2(\bar{c}_1 - A)}{F_2(\bar{c}_1 - A)} \limsup \left( \frac{\tilde{F}_1'(\phi(c)) + (\bar{c}_1 - \phi(c)) \left( \frac{-\tilde{F}_1'(s)}{f_1(s)} \right)_{s=\phi(c)}'}{f_1'(\phi(c))} \right)
\]

\[
\leq \frac{f_2(\bar{c}_1 - A)}{F_2(\bar{c}_1 - A)} \left( \limsup \left( \frac{\tilde{F}_1'(\phi(c))}{f_1'(\phi(c))} \right) + \limsup \left( (\bar{c}_1 - \phi(c)) \left( \frac{-\tilde{F}_1'(s)}{f_1(s)} \right)_{s=\phi(c)}' \right) \right)
\]

\[
= \frac{f_2(\bar{c}_1 - A)}{F_2(\bar{c}_1 - A)} \limsup \left( (\bar{c}_1 - \phi(c)) \left( \frac{-\tilde{F}_1'(s)}{f_1(s)} \right)_{s=\phi(c)}' \right)
\]

But, by Lemma 24 \( \limsup \left( \frac{-\tilde{F}_1'(s)}{f_1(s)} \right)_{s=\phi(c)}' \in [0, 1] \). The term \((\bar{c}_1 - \phi(c))\) is bounded as well. Thus,

\[
1 \leq \frac{f_2(\bar{c}_1 - A)}{F_2(\bar{c}_1 - A)} \limsup (\bar{c}_1 - \phi(c)) \limsup \left( \frac{-\tilde{F}_2'(s)}{f_2'(s)} \right)_{s=\phi(c)}'
\]

\[
\leq \frac{f_2(\bar{c}_1 - A)}{F_2(\bar{c}_1 - A)} \limsup (\bar{c}_1 - \phi(c))
\]

\[
= 0,
\]

a contradiction. \( \blacksquare \)

**Lemma 28** \( \limsup \phi'(c) > 0 \).
**Proof** If \( \limsup \phi' (c) = 0 \) then \( \phi' (c) \to 0 \). So, for any small \( t \), there is a last \( c(t) \) at which \( \phi' (c) = t \) (this is well defined since \( \phi \) is continuously differentiable and \([0, c_1 - A] \) is compact). But, by a change of variables,

\[
S_1 (\phi (c(t))) = \int_{\phi(c(t))}^{\hat{c}_1} \tilde{F}_2 (\psi (s)) \, ds \\
= \int_{c(t)}^{\hat{c}_1 - A} \tilde{F}_2 (s) \phi' (s) \, ds, \\
< t (\hat{c}_1 - A - c(t)),
\]

since \( \phi' (s) < t \), and \( \tilde{F}_2 < 1 \).

Thus,

\[
\frac{1}{S_1 (\phi (c(t)))} \frac{f_2}{f_1} (c(t)) > \frac{t}{t (\hat{c}_1 - A - c(t))} \frac{f_2}{f_1} (c(t)) \\
> \frac{1}{\tilde{F}_2 (\hat{c}_1 - A) t (\hat{c}_1 - A - c(t))}.
\]

This diverges as \( t \to 0 \) and \( c(t) \to \hat{c}_1 - A \). But then by Proposition 7 for small \( t \), \( \phi'' (c(t)) > 0 \), contradicting that \( c(t) \) was the last moment at which \( \phi' = t \). ■

Since we have ruled out \( \limsup \phi' (c) \in [0, \infty) \), we have that \( \limsup \phi' (c) = \infty \), proving the first part of Lemma 16. To conclude from this that \( \phi' (c) \to \infty \), we would need to tie down \( \liminf \phi' (c) \). This turns out to be tricky, but enough for our purposes is given by the following.

**Lemma 29** There is \( \hat{c} < \hat{c}_1 - A \) such that for all \( c > \hat{c} \), \( \phi' (c) > 1 \).

**Proof** From Proposition 7 we have that

\[
\phi''_{FP} (c) \geq_s \left( \frac{1}{S_1 (\phi_{FP} (c))} \frac{f_2}{f_1} (c) - 2 \right) \left( \frac{f_1}{f_2} (\phi_{FP} (c)) - \frac{f_2}{f_1} (c) \right) + \left( \frac{f_2'}{f_2} (c) - \frac{f_1'}{f_1} (\phi_{FP} (c)) \right),
\]

If \( \phi' (c) = 1 \), but \( c \) is close to \( \hat{c}_1 - A \), \( \phi_{FP} > \gamma \frac{\gamma'}{\frac{\gamma}{\gamma'}} \frac{\gamma}{\gamma'} \) by Lemma 11 and by Lemma 6 \( \gamma \frac{\gamma}{\gamma'} \geq \gamma \frac{\gamma'}{\gamma'} \) thus the last two terms are positive

\[
\frac{\phi'' (c)}{\phi' (c)} \geq_s \left( \frac{1}{S_1 (\phi (c))} \frac{f(c)}{f_2 (c)} - 2 \right).
\]
But,

\[
S_1 (\phi (c)) \frac{f_2 (c)}{F_2^2 (c)} = \int_{\phi (c)}^{\tilde{c}_1} F (\psi (s)) \, ds \frac{f_2 (c)}{F_2^2 (c)} < \int_{\phi (c)}^{\tilde{c}_1} 1 \, ds \frac{f_2 (c)}{F_2^2 (c)} = (\tilde{c}_1 - \phi (c)) \frac{f_2 (c)}{F_2^2 (c)}
\]

which goes to 0. So, there is \( c^* \) such that for \( c > c^* \), any point where \( \phi' = 1 \) has \( \phi'' > 0 \), and so there is at most one (upward) crossing of \( \phi' = 1 \) after that. Let \( \hat{c} \) be that upward crossing if it exists, and \( \hat{c} = c^* \) otherwise.

**Lemma 30** At any point in \([\underline{c}_1, \hat{c}_1]\) where \( f' \neq 0 \),

\[
\frac{(\log f)''}{(\log F)''} = \frac{W_F^2 (1 - W_f)}{(1 - W_F)}.
\]

If \( f (\tilde{c}_1) = 0 \), and \( \frac{(\log f)''}{(\log F)''} (\tilde{c}_1) \) is well defined, then \( \frac{(\log f)''}{(\log F)''} (\tilde{c}_1) = \frac{W_F^2 (1 - W_f)}{(1 - W_F)} (\tilde{c}_1) \) regardless of \( \frac{f'}{F} (\tilde{c}_1) \).

**Proof** Note that

\[
\frac{(\log f)''}{(\log F)''} = \left( \frac{-f'}{F} \right)' = \frac{-f'' + \left( \frac{f'}{F} \right)^2}{\left( \frac{f'}{F} + \frac{f''}{F} \right)}.
\]  \hspace{1cm} (42)

On \([0, \tilde{c}_1], \frac{f}{F} \in (0, \infty), \) and \( \frac{f'}{F} \in (-\infty, \infty). \) So, where \( \frac{f'}{F} \neq 0, \) we have

\[
\frac{(\log f)''}{(\log F)''} = -\left( \frac{f'}{F} \right)^2 \left( \frac{f''}{(f')^2} - 1 \right) = \frac{W_F^2 (1 - W_f)}{(1 - W_F)} \cdot \hspace{1cm} (43)
\]

Assume that \( f (\tilde{c}_1) = 0. \) Then, on some interval \((\tilde{c}, \tilde{c}_1), f' \neq 0 \) (since, by log-concavity, \( f' \) crosses 0 on at most one point or interval). Hence, on this interval, \( \frac{(\log f)''}{(\log F)''} = \frac{W_F^2 (1 - W_f)}{(1 - W_F)} \). Since \( \frac{(\log f)''}{(\log F)''} (\tilde{c}_1) \) is well-defined by assumption, and since both sides are continuous on \((\tilde{c}, \tilde{c}_1), \) the claim follows.
Proof of Lemma 20 Assume first that \( f(\tilde{c}_1) = 0 \). Then, from Lemma 2 with \( h = f \), we have, with a little rearrangement,

\[
1 - W_f(\tilde{c}_1) = \frac{1}{W_F(\tilde{c}_1)} - 1
\]

or

\[
\frac{W_F(\tilde{c}_1) (1 - W_f(\tilde{c}_1))}{1 - W_f(\tilde{c}_1)} = 1
\]

and thus

\[
\frac{W_F(\tilde{c}_1) (1 - W_f(\tilde{c}_1))}{1 - W_f(\tilde{c}_1)} = W_F(\tilde{c}_1).
\]

If \( f'(\tilde{c}_1) \neq 0 \), we are thus done, by Lemma 30.

On the other hand, assume that \( f(\tilde{c}_1) > 0 \). Then, \( W_F(\tilde{c}_1) = 0 \), and so

\[
\frac{W_F(\tilde{c}_1) (1 - W_f(\tilde{c}_1))}{1 - W_f(\tilde{c}_1)} = W_F(\tilde{c}_1) \text{ holds again (recall that we assume } W_f(\tilde{c}_1) \text{ to be finite).} \]

9.5 Proofs for Section 7

Proof of Proposition 9 Since \( \phi'(c) \geq 1 \),

\[
\frac{\partial}{\partial c} \ln \frac{\bar{F}(\phi(c))}{F(c)} = \frac{\partial}{\partial s} \ln \bar{F}(s) \big|_{s=\phi(c)} - \frac{\partial}{\partial s} \ln \bar{F}(s) \big|_{s=c}
\]

\[
\leq 0
\]

where the first inequality holds since \( \phi'(c) \geq 1 \), and \( \frac{\partial}{\partial s} \ln \bar{F}(s) \big|_{s=\phi(c)} < 0 \) since \( \bar{F} \) is decreasing, and the second inequality holds since \( \bar{F} \) is log-concave and decreasing and \( \phi(c) \geq c \). Thus, \( \frac{\bar{F}(\phi(c))}{F(c)} \) is decreasing in \( c \).

By Cauchy’s theorem for all \( c \in [0,1-A) \), there is \( \xi_c \in [c,1-A) \) such that

\[
\int_c^{1-A} \frac{\bar{F}(\phi(s))ds}{\bar{F}(s)} = \frac{\bar{F}(\phi(\xi_c))}{\bar{F}(\xi_c)} \leq \frac{\bar{F}(\phi(c))}{\bar{F}(c)}
\]

or, rearranging,

\[
\frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} \geq \frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\bar{F}(\phi(c))}.
\]

Adding \( c \) to each side yields

\[
\beta_s(c) = c + \frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} \geq c + \frac{\int_c^{1-A} \bar{F}(\phi(s))ds}{\bar{F}(\phi(c))} = \beta_2(c),
\]

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where $\beta_s(c)$ is the symmetric equilibrium bid function. Similarly,

$$
\beta_s(c) = c + \frac{\int_c^1 \bar{F}(s)ds}{\bar{F}(c)} \leq c + \frac{\int_c^1 \bar{F}(\psi(s))ds}{\bar{F}(\psi(c))} = \beta_1(c). \quad \blacksquare
$$
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